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Gabor Analysis on Wiener Amalgams

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Abstract

A general summability method, the so-called θ-summability, is considered for Gabor series. Under suitable conditions on θ we prove that this summation method of the Gabor expansion of f converges to f in Wiener amalgam norms, and in particular with respect to L_p-norms, for functions f from the corresponding spaces, as well as almost everywhere. Some inequalities for the maximal operator of the θ-means of the Gabor expansion are obtained. The analogous statements for the partial sums of Gabor series are also given. The classical Hardy-Littlewood inequality and the Marcinkiewicz multiplier theorem is shown to be valid in the context of Gabor series.

Key words and phrases : Wiener amalgam spaces, Herz spaces, θ-summability, Gabor expansions, Gabor frames, time-frequency analysis, Walnut representation, Hardy-Littlewood inequality, multipliers

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1 Introduction

It is known for some well-known summability methods, such as Fejér, Riesz, Weierstrass, Abel, etc., that the corresponding means $\sigma_n f$ of the Fourier series of f converge to f uniformly as $n \to \infty$ if f is continuous, and in L_p norm if $f \in L_p(\mathbb{T}^d)$ for some $1 \leq p < \infty$ (see e.g. Fejér [12], Zygmund [29], Butzer and Nessel [2], Stein and Weiss [24] or Trigub and Belinsky [25]).

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A general method of summation, the so-called θ-summation, which is generated by a single function θ, is intensively studied in the literature (see e.g. Butzer and Nessel [2], Trigub and Belinsky [25] and Weisz [28] and the references therein). If the Fourier transform of θ is integrable, then the preceding convergence results hold for the θ-summation, too (see Butzer and Nessel [2], Trigub and Belinsky [25] or Feichtinger and Weisz [9]). We proved in [10] that $\sigma_n f(x) \to f(x)$ a.e. (more exactly, at each p-Lebesgue point of f) for all $f \in L_p(\mathbb{T}^d)$, whenever $\hat{\theta}$ is in the homogeneous Herz space $\dot{E}_{p'}$, $1 \leq p < \infty$, $1/p + 1/p' = 1$.

In this paper we will extend these results to the summation of Gabor expansions $\sum_{k,n\in\mathbb{Z}^d} \langle f, M_{\beta n} T_{\alpha k} g\rangle M_{\beta n} T_{\alpha k} \gamma$ and to Wiener amalgam spaces $W(L_p, \ell_q)(\mathbb{R}^d)$, where $\alpha, \beta > 0$, $g, \gamma \in W(L_\infty, \ell_1)(\mathbb{R}^d)$ and M denotes the modulation operator and T the translation operator. Gröchenig, Heil and Okoudjou [17, 18] made use of the Banach space $s_{p,q}$ of complex sequences and proved that the coefficient operator $C_g : f \mapsto (\langle f, M_{\beta n} T_{\alpha k} g\rangle)_{k,n\in\mathbb{Z}^d}$ is bounded from $W(L_p, \ell_q)(\mathbb{R}^d)$ to $s_{p,q}$ and the reconstruction operator R_γ is bounded from $s_{p,q}$ to $W(L_p, \ell_q)(\mathbb{R}^d)$. In the case where $p = \infty$ or $q = \infty$ they obtained weak convergence of R_γ. By taking a closed subspace of $W(L_p, \ell_\infty)$, namely the space $W(L_p, c_0)$, we will prove strong type results in the endpoint case, too. We need this result to verify later the uniform convergence of Gabor series.

For $c = (c_{k,n})_{k,n\in\mathbb{Z}^d} \in s_{p,q}$ we investigate the θ-means $\sigma^\theta_{K,N} c$ of the Gabor series with coefficients c and get that $\sigma^\theta_{K,N} c \to R_\gamma c$ in $W(L_p, \ell_q)(\mathbb{R}^d)$ norm as $K, N \to \infty$ $(1 \leq p < \infty)$, whenever $\hat{\theta} \in L_1(\mathbb{R}^d)$. If g defines a Gabor frame $\mathcal{G}(g, \alpha, \beta)$ with dual frame $\mathcal{G}(\gamma, \alpha, \beta)$, then the θ-means $\sigma^\theta_{K,N} f$ of the Gabor expansion of $f \in W(L_p, \ell_q)(\mathbb{R}^d)$ converge to f in norm. Moreover, if f, g, and γ are continuous and $p = \infty$, we obtain uniform convergence.

We will show similar results for the pointwise convergence. If γ has compact support and $c \in s_{p,q}$ then $\sigma^\theta_{K,N} c \to R_\gamma c$ a.e., whenever $\hat{\theta} \in \dot{E}_{r'}(\mathbb{R}^d)$, $1 \leq r \leq p < \infty$, $1/r + 1/r' = 1$. If γ has no compact support, then the convergence holds for $1 \leq r < p < \infty$. If in addition $\mathcal{G}(g, \alpha, \beta)$ is a Gabor frame with dual frame $\mathcal{G}(\gamma, \alpha, \beta)$, then $\sigma^\theta_{K,N} f \to f$ a.e., where $f \in W(L_p, \ell_q)(\mathbb{R}^d)$. Moreover, the maximal operator of the θ-means of the Gabor expansion is bounded on $W(L_p, \ell_q)(\mathbb{R}^d)$. Analogous results are obtained for the partial sums of Gabor series as well. Finally, the Hardy-Littlewood inequality and Marcinkiewicz multiplier theorem are generalized for Gabor expansions.

2 Wiener amalgam spaces

Let us fix $d \geq 1$, $d \in \mathbb{N}$. For a set $\mathbb{Y} \neq \emptyset$ let $\mathbb{Y}^d$ be its Cartesian product $\mathbb{Y} \times \ldots \times \mathbb{Y}$ taken with itself d-times. For $x = (x_1, \ldots, x_d) \in \mathbb{R}^d$ and $u = (u_1, \ldots, u_d) \in \mathbb{R}^d$

set

$$u \cdot x := \sum_{k=1}^{d} u_k x_k \qquad \text{and} \qquad |x| := \max_{k=1,\ldots,d} |x_k|.$$

The ℓ_p $(1 \le p \le \infty)$ space consists of all complex sequences $a = (a_k)_{k\in\mathbb{Z}^d}$ for which

$$\|a\|_{\ell_p} := \Big(\sum_{k\in\mathbb{Z}^d} |a_k|^p \Big)^{1/p} < \infty$$

with the usual modification if $p = \infty$. The set of sequences (a_k) with the property $\lim_{|k|\to\infty} a_k = 0$ is denoted by c_0 and it is equipped with the ℓ_∞ norm.

We briefly write L_p or $L_p(\mathbb{R}^d)$ instead of $L_p(\mathbb{R}^d, \lambda)$ space equipped with the norm (or quasi-norm) $\|f\|_p := (\int_{\mathbb{R}^d} |f|^p \, d\lambda)^{1/p}$ $(0 < p \le \infty)$, where λ is the Lebesgue measure. The space of continuous functions with the supremum norm is denoted by $C(\mathbb{R}^d)$ and we will use $C_0(\mathbb{R}^d)$ for the space of continuous functions vanishing at infinity. $C_c(\mathbb{R}^d)$ denotes the space of continuous functions having compact support.

Translation and *modulation* of a function f are defined, respectively, by

$$T_x f(t) := f(t-x) \quad \text{and} \quad M_\omega f(t) := e^{2\pi\imath\omega\cdot t} f(t) \qquad (x, \omega \in \mathbb{R}^d).$$

For a set H we use the notation $T_x H := H - x$. The *Fourier transform* of $f \in L_1(\mathbb{R}^d)$ is

$$\mathcal{F} f(x) := \hat{f}(x) := \int_{\mathbb{R}^d} f(t) e^{-2\pi\imath x\cdot t} \, dt \qquad (x \in \mathbb{R}^d),$$

where $\imath = \sqrt{-1}$.

Let Q and Q_α denote the cubes

$$Q = [0,1)^d, \qquad Q_\alpha = [0,\alpha)^d \qquad (\alpha > 0).$$

A measurable function f belongs to the *Wiener amalgam space* $W(L_p, \ell_q)(\mathbb{R}^d)$ $(1 \le p, q \le \infty)$ if

$$\|f\|_{W(L_p,\ell_q)} := \Big(\sum_{k\in\mathbb{Z}^d} \|f(\cdot + k)\|_{L_p(Q)}^q \Big)^{1/q} = \Big(\sum_{k\in\mathbb{Z}^d} \|f \cdot T_k \mathbf{1}_Q\|_p^q \Big)^{1/q} < \infty,$$

with the obvious modification for $q = \infty$. It is easy to show that the norm

$$\|f\| := \Big(\sum_{k\in\mathbb{Z}^d} \|f \cdot T_{\alpha k} \mathbf{1}_{Q_\alpha}\|_p^q \Big)^{1/q} < \infty$$

is an equivalent norm on $W(L_p, \ell_q)(\mathbb{R}^d)$. The closed subspace of $W(L_\infty, \ell_q)(\mathbb{R}^d)$ containing continuous functions is denoted by $W(C, \ell_q)(\mathbb{R}^d)$ $(1 \le q \le \infty)$.

$W(L_p, c_0)(\mathbb{R}^d)$ is defined analogously $(1 \le p \le \infty)$. The space $W(C, \ell_1)(\mathbb{R}^d)$ is called *Wiener algebra*.

It is easy to see that $W(L_p, \ell_p)(\mathbb{R}^d) = L_p(\mathbb{R}^d)$,

$$W(L_{p_1}, \ell_q)(\mathbb{R}^d) \hookleftarrow W(L_{p_2}, \ell_q)(\mathbb{R}^d) \qquad (p_1 \le p_2)$$

and

$$W(L_p, \ell_{q_1})(\mathbb{R}^d) \hookrightarrow W(L_p, \ell_{q_2})(\mathbb{R}^d) \qquad (q_1 \le q_2),$$

$(1 \le p_1, p_2, q_1, q_2 \le \infty)$. Thus,

$$W(L_\infty, \ell_1)(\mathbb{R}^d) \subset L_p(\mathbb{R}^d) \subset W(L_1, \ell_\infty)(\mathbb{R}^d) \qquad (1 \le p \le \infty).$$

In this paper the constants C and C_p may vary from line to line and the constants C_p are dependent only on p.

3 Reconstruction and coefficient operators

Given a window $g \in L_2(\mathbb{R}^d)$ and $\alpha, \beta > 0$, we say that the collection

$$\mathcal{G}(g, \alpha, \beta) := \{M_{\beta n} T_{\alpha k} g : k, n \in \mathbb{Z}^d\}$$

is a *Gabor frame* for $L_2(\mathbb{R}^d)$ if there exist constants $A, B > 0$ such that

$$A\|f\|_2^2 \le \sum_{k,n \in \mathbb{Z}^d} |\langle f, M_{\beta n} T_{\alpha k} g\rangle|^2 \le B\|f\|_2^2$$

for all $f \in L_2(\mathbb{R}^d)$. In this case there exists a *dual window* $\gamma \in L_2(\mathbb{R}^d)$ such that $\mathcal{G}(\gamma, \alpha, \beta)$ is also a Gabor frame for $L_2(\mathbb{R}^d)$ and

$$f = \sum_{k,n \in \mathbb{Z}^d} \langle f, M_{\beta n} T_{\alpha k} g\rangle M_{\beta n} T_{\alpha k} \gamma = \sum_{k,n \in \mathbb{Z}^d} \langle f, M_{\beta n} T_{\alpha k} \gamma\rangle M_{\beta n} T_{\alpha k} g \qquad (1)$$

for all $f \in L_2(\mathbb{R}^d)$. This series converges unconditionally in $L_2(\mathbb{R}^d)$ and the ℓ_2 norm of the Gabor coefficients $(\langle f, M_{\beta n} T_{\alpha k} \gamma\rangle)$ is an equivalent norm on $L_2(\mathbb{R}^d)$. For more details we refer to Daubechies [4] or Gröchenig [16].

Under some stronger condition on g and γ, (1) is also valid for other function spaces. If g, γ is in the Feichtinger's algebra, then (1) holds for modulation spaces (see Feichtinger and Zimmermann [11] and Gröchenig [16]) and if $g, \gamma \in W(L_\infty, \ell_1)$, then for L_p and amalgam spaces (Gröchenig, Heil, and Okoudjou [17, 18]). In the last case the convergence is conditional; first we sum over n and then over k. Summing over n in the first sum in (1), we obtain formally the trigonometric series

$$m_k(x) = \sum_{n \in \mathbb{Z}^d} \langle f, M_{\beta n} T_{\alpha k} g\rangle e^{2\pi \imath \beta n \cdot x}$$

with period $1/\beta$. Then (1) reads as

$$f = \sum_{k \in \mathbb{Z}^d} m_k T_{\alpha k} \gamma.$$

Let us introduce the space $s_{p,q}$ as in Gröchenig, Heil, and Okoudjou [18]. The n^{th} Fourier coefficient of a $1/\beta$ periodic function $h \in L_1(Q_{1/\beta})$ is given by

$$\hat{h}(n) := \beta^d \int_{Q_{1/\beta}} h(t) e^{-2\pi\imath\beta n \cdot t} \, dt \qquad (n \in \mathbb{Z}^d).$$

A sequence $c = (c_{k,n})_{k,n \in \mathbb{Z}^d}$ of complex numbers is in $s_{p,q}$ $(1 \le p, q \le \infty)$ if there exist $1/\beta$ periodic functions $m_k \in L_p(Q_{1/\beta})$ such that

$$\hat{m}_k(n) = c_{k,n}, \qquad k, n \in \mathbb{Z}^d$$

and

$$\|c\|_{s_{p,q}} := \Big(\sum_{k \in \mathbb{Z}^d} \|m_k\|_{L_p(Q_{1/\beta})}^q \Big)^{1/q} < \infty$$

with the usual modification for $q = \infty$. Note that the functions m_k are unique. If $1 < p < \infty$, then m_k can be written as the Fourier series

$$m_k(x) = \sum_{n \in \mathbb{Z}^d} c_{k,n} e^{2\pi\imath\beta n \cdot x}$$

in the sense that the rectangular partial sums converge to m_k in the norm of $L_p(Q_{1/\beta})$ (cf. Zygmund [29] or Weisz [28]).

The closed subspace $s_{p,q,0}$ contains all elements of $s_{p,q}$ for which

$$\lim_{k \to \infty} \|m_k\|_{L_p(Q_{1/\beta})} = 0.$$

Of course, $s_{p,q} = s_{p,q,0}$ if $1 \le p \le \infty$ and $1 \le q < \infty$. Similarly, let $\ell_{q,0} := \ell_q$ if $1 \le q < \infty$ and $\ell_{\infty,0} := c_0$.

The following two theorems are proved by Gröchenig, Heil, and Okoudjou [17, 18] for Wiener amalgam spaces $W(L_p, \ell_q)(\mathbb{R}^d)$ if $1 \le p, q \le \infty$ and by Balan and Daubechies [1] for $W(L_2, \ell_\infty)(\mathbb{R}^d)$. They obtained weak convergence for $p = \infty$ and/or $q = \infty$. For this endpoint case we verify here strong type theorems.

Theorem 1 *Assume that $\gamma \in W(L_\infty, \ell_1)(\mathbb{R}^d)$ and $c \in s_{p,q}$ for some $1 \le p, q \le \infty$. Then the reconstruction operator*

$$R_\gamma c := \sum_{k \in \mathbb{Z}^d} m_k T_{\alpha k} \gamma \tag{2}$$

converges unconditionally in $W(L_p,\ell_q)(\mathbb{R}^d)$ norm if $1 \le q < \infty$ and unconditionally in the w^* topology of $W(L_p,\ell_\infty)(\mathbb{R}^d)$ if $q=\infty$. If $c \in s_{p,\infty,0}$, then the convergence holds unconditionally in $W(L_p,\ell_\infty)(\mathbb{R}^d)$ norm. Moreover, R_γ is bounded from $s_{p,q}$ to $W(L_p,\ell_q)(\mathbb{R}^d)$ and from $s_{p,\infty,0}$ to $W(L_p,\ell_{\infty,0})(\mathbb{R}^d)$ and

$$\|R_\gamma c\|_{W(L_p,\ell_q)} \le C\|\gamma\|_{W(L_\infty,\ell_1)}\|c\|_{s_{p,q}}. \tag{3}$$

If $q \le p$ and $c \in s_{p,q,0}$, then the sum in (2) converges unconditionally a.e.

Proof. Except the results concerning the space $s_{p,\infty,0}$ and the a.e. convergence, Theorem 1 was proved in Gröchenig, Heil, and Okoudjou [18] with the help of the inequality

$$\begin{aligned} &\Big|\Big\langle \sum_{k\in\mathbb{Z}^d} m_k T_{\alpha k}\gamma, h\Big\rangle\Big| \\ &\le C\sum_{l\in\mathbb{Z}^d}\sum_{k\in\mathbb{Z}^d} \|\gamma T_{\alpha l}\mathbf{1}_{Q_\alpha}\|_\infty \|m_k\|_{L_p(Q_{1/\beta})}\|hT_{\alpha k+\alpha l}\mathbf{1}_{Q_\alpha}\|_{p'} \qquad (4)\\ &\le C\|\gamma\|_{W(L_\infty,\ell_1)}\Big(\sum_{k\in\mathbb{Z}^d}\|m_k\|^q_{L_p(Q_{1/\beta})}\Big)^{1/q}\|h\|_{W(L_{p'},\ell_{q'})}, \end{aligned}$$

where $h \in W(L_{p'},\ell_{q'})(\mathbb{R}^d)$ and p' denotes the dual index to p. Note that the dual space of $W(L_p,\ell_q)(\mathbb{R}^d)$ is $W(L_{p'},\ell_{q'})(\mathbb{R}^d)$.

From this inequality we can see the unconditional convergence in $W(L_p,\ell_\infty)(\mathbb{R}^d)$ norm, too, if $c \in s_{p,\infty,0}$. Since, for a fixed $k \in \mathbb{Z}^d$, $T_{\alpha k}\gamma \in W(L_\infty,\ell_1)(\mathbb{R}^d)$, we have

$$\|m_k T_{\alpha k}\gamma\|_{L_p(T_jQ)} \le C\sup_{T_jQ}|T_{\alpha k}\gamma|\ \|m_k\|_{L_p(Q_{1/\beta})} \to 0 \qquad \text{as} \qquad j\to\infty,$$

and so $m_kT_{\alpha k}\gamma \in W(L_p,c_0)(\mathbb{R}^d)$. This implies $R_\gamma c \in W(L_p,c_0)(\mathbb{R}^d)$ because $W(L_p,c_0)(\mathbb{R}^d)$ is complete.

For the almost everywhere convergence of the sum in (2) observe that $\ell_q \hookrightarrow \ell_p$ if $q \le p$ and so

$$\Big(\sum_{k\in\mathbb{Z}^d}\|m_k\|^p_{L_p(Q_{1/\beta})}\Big)^{1/p} \le \Big(\sum_{k\in\mathbb{Z}^d}\|m_k\|^q_{L_p(Q_{1/\beta})}\Big)^{1/q} = \|c\|_{s_{p,q,0}} < \infty.$$

Hence,

$$\sum_{k\in\mathbb{Z}^d}\int_{Q_{1/\beta}}|m_k|^p\,d\lambda = \int_{Q_{1/\beta}}\sum_{k\in\mathbb{Z}^d}|m_k|^p\,d\lambda < \infty.$$

This implies that $\sum_{k\in\mathbb{Z}^d}|m_k|^p$ is a.e. finite and

$$m_k \to 0 \qquad \text{a.e. as } k\to\infty. \tag{5}$$

Consequently,

$$|R_\gamma c - \sum_{|k|\le K} m_k T_{\alpha k}\gamma| \le \sum_{|k|>K} |m_k T_{\alpha k}\gamma| \le \|\gamma\|_{W(L_\infty,\ell_1)} \sup_{|k|>K} |m_k| \to 0$$

as $K \to \infty$. ■

Note that if γ has compact support, then the sum in (2) is finite for every fixed x.

Theorem 2 *If $g \in W(L_\infty, \ell_1)(\mathbb{R}^d)$ and $f \in W(L_p, \ell_q)(\mathbb{R}^d)$ for some $1 \le p, q \le \infty$ then the coefficient operator*

$$C_g f := (\langle f, M_{\beta n} T_{\alpha k} g\rangle)_{k,n\in\mathbb{Z}^d}$$

is bounded from $W(L_p, \ell_q)(\mathbb{R}^d)$ to $s_{p,q}$ and from $W(L_p, \ell_{\infty,0})(\mathbb{R}^d)$ to $s_{p,\infty,0}$ and

$$\|C_g f\|_{s_{p,q}} \le C\|g\|_{W(L_\infty,\ell_1)}\|f\|_{W(L_p,\ell_q)}. \tag{6}$$

Moreover, there exist unique functions $m_k \in L_p(Q_{1/\beta})$ which satisfy $\hat{m}_k(n) = C_g f(k,n)$ for all $k, n \in \mathbb{Z}^d$ and these are given explicitly by

$$m_k(x) = \beta^{-d} \sum_{n\in\mathbb{Z}^d} (f \cdot T_{\alpha k}\overline{g})(x - n/\beta) \tag{7}$$

with unconditional convergence in $L_p(Q_{1/\beta})$.

Proof. Except the results concerning the space $s_{p,\infty,0}$ and the norm convergence in (7) for $p = \infty$, the theorem was proved in Gröchenig, Heil, and Okoudjou [18]. The norm convergence in (7) follows from

$$\begin{aligned}
\|m_k\|_{L_p(Q_{1/\beta})} &\le \beta^{-d} \sum_{n\in\mathbb{Z}^d} \|(f \cdot T_{\alpha k}\overline{g})(\cdot - n/\beta)\|_{L_p(Q_{1/\beta})} \\
&= \beta^{-d} \sum_{n\in\mathbb{Z}^d} \|f \cdot T_{\alpha k} g\|_{L_p(T_{n/\beta}Q_{1/\beta})} \\
&\le \beta^{-d} \sum_{n\in\mathbb{Z}^d} \sup_{T_{n/\beta}Q_{1/\beta}} |T_{\alpha k} g| \|f\|_{L_p(T_{n/\beta}Q_{1/\beta})} \qquad (8) \\
&\le C\beta^{-d} \sum_{n\in\mathbb{Z}^d} \sup_{T_{n/\beta}Q_{1/\beta}} |T_{\alpha k} g| \|f\|_{W(L_p,\ell_q)} \\
&\le C\beta^{-d} \|g\|_{W(L_\infty,\ell_1)} \| \|f\|_{W(L_p,\ell_q)}
\end{aligned}$$

for all $1 \le p \le \infty$. We must show that if $f \in W(L_p, c_0)(\mathbb{R}^d)$, then $C_g f \in s_{p,\infty,0}$, i.e., $\lim_{k\to\infty} \|m_k\|_{L_p(Q_{1/\beta})} = 0$. In this case if n is large enough in (8), say

$|n| \geq N$, then $\|f\|_{L_p(T_{n/\beta}Q_{1/\beta})} < \epsilon$ and so

$$\begin{aligned}\beta^{-d} \sum_{|n|\geq N} \sup_{T_{n/\beta}Q_{1/\beta}} |T_{\alpha k} g| \|f\|_{L_p(T_{n/\beta}Q_{1/\beta})} &\leq \epsilon \beta^{-d} \sum_{|n|\geq N} \sup_{T_{n/\beta}Q} |T_{\alpha k} g| \\ &\leq \epsilon \beta^{-d} \|g\|_{W(L_\infty,\ell_1)}.\end{aligned}$$

On the other hand, if $|n| < N$ in (8), then there exists a number K_k such that

$$\begin{aligned}&\beta^{-d} \sum_{|n|<N} \sup_{T_{n/\beta}Q_{1/\beta}} |T_{\alpha k} g| \|f\|_{L_p(T_{n/\beta}Q_{1/\beta})} \\ &\qquad \leq C\beta^{-d} \|f\|_{W(L_p,c_0)} \sum_{|n|<N} \sup_{T_{\alpha k+n/\beta}Q_{1/\beta}} |g| \\ &\qquad \leq C\beta^{-d} \|f\|_{W(L_p,c_0)} \sum_{|j|\geq K_k} \sup_{T_j Q} |g|.\end{aligned}$$

It is easy to see that $K_k \to \infty$ as $k \to \infty$ and then

$$\sum_{|j|\geq K_k} \sup_{T_j Q} |g| \to 0 \qquad \text{as} \qquad k \to \infty.$$

This means that

$$\beta^{-d} \sum_{|n|<N} \sup_{T_{n/\beta}Q_{1/\beta}} |T_{\alpha k} g| \|f\|_{L_p(T_{n/\beta}Q_{1/\beta})} < \epsilon$$

if k is large enough; thus, $\lim_{k\to\infty} \|m_k\|_{L_p(Q_{1/\beta})} = 0$. ■

The *Gabor frame operator* is defined formally by

$$S_{g,\gamma} f := \sum_{k,n\in\mathbb{Z}^d} \langle f, M_{\beta n} T_{\alpha k} g\rangle M_{\beta n} T_{\alpha k} \gamma.$$

If we give the meaning $S_{g,\gamma} f := R_\gamma C_g f$ to this definition, then we obtain the following.

Corollary 1 *If $g, \gamma \in W(L_\infty, \ell_1)(\mathbb{R}^d)$, then $S_{g,\gamma}$ is bounded on $W(L_p, \ell_q)(\mathbb{R}^d)$ and on $W(L_p, \ell_{\infty,0})(\mathbb{R}^d)$ $(1 \leq p, q \leq \infty)$, and*

$$\|S_{g,\gamma} f\|_{W(L_p,\ell_q)} \leq C \|g\|_{W(L_\infty,\ell_1)} \|\gamma\|_{W(L_\infty,\ell_1)} \|f\|_{W(L_p,\ell_q)}.$$

The following two results are due to Gröchenig, Heil, and Okoudjou [17, 18].

Theorem 3 *If $g,\gamma \in W(L_\infty,\ell_1)(\mathbb{R}^d)$ and $f \in W(L_p,\ell_q)(\mathbb{R}^d)$ for some $1 \leq p,q \leq \infty$, then the Walnut representation*

$$R_\gamma C_g f = \beta^{-d} \sum_{n\in\mathbb{Z}^d} G_n \cdot T_{n/\beta} f \tag{9}$$

holds with absolute convergence in $W(L_p,\ell_q)(\mathbb{R}^d)$, where

$$G_n(x) := \sum_{k\in\mathbb{Z}^d} \overline{g(x-n/\beta-\alpha k)}\gamma(x-\alpha k). \tag{10}$$

Corollary 2 *Assume that $g,\gamma \in W(L_\infty,\ell_1)(\mathbb{R}^d)$ such that $\mathcal{G}(g,\alpha,\beta)$ is a Gabor frame for $L_2(\mathbb{R}^d)$ with dual frame $\mathcal{G}(\gamma,\alpha,\beta)$. If $f \in W(L_p,\ell_q)(\mathbb{R}^d)$ for some $1 \leq p,q \leq \infty$, then $R_\gamma C_g f = f$ and we have the norm equivalence $\|f\|_{W(L_p,\ell_q)} \sim \|C_g f\|_{s_{p,q}}$.*

4 Norm convergence of Gabor expansions

It is known that the rectangular partial sums of the multi-dimensional Fourier series

$$\sum_{n\in\mathbb{Z}^d} \hat{h}(n) e^{2\pi\imath\beta n\cdot x}$$

of $h \in L_p(Q_{1/\beta})$ $(1 < p < \infty)$ converge to h in $L_p(Q_{1/\beta})$ norm (cf. Zygmund [29] or Weisz [28]). Moreover, according to one of the deepest result in harmonic analysis, the square partial sums of the Fourier series converge a.e. to $h \in L_p(Q_{1/\beta})$ $(1 < p < \infty)$ (see Carleson [3], Hunt [20] and, in the more-dimensional case, Fefferman [7], and also Grafakos [14]), i.e.,

$$S_N h \to h \qquad \text{in } L_p(Q_{1/\beta}) \text{ norm and a.e. as } N \to \infty, \tag{11}$$

where

$$S_N h(x) := \sum_{|n|\leq N} \hat{h}(n) e^{2\pi\imath\beta n\cdot x} \qquad (N \in \mathbb{N}).$$

Using these theorems, similar convergence results will be proven for Gabor expansions in this and the next sections.

For $c \in s_{p,q}$ $(1 \leq p,q \leq \infty)$ and $\gamma \in W(L_\infty,\ell_1)$ let

$$S_{\gamma,K,N} c := S_{K,N} c := \sum_{|k|\leq K} \sum_{|n|\leq N} c_{k,n} M_{\beta n} T_{\alpha k}\gamma \qquad (K,N \in \mathbb{N}).$$

Then $S_{K,\infty} c$ means formally

$$S_{K,\infty} c(x) = \sum_{|k|\leq K} \Big(\sum_{n\in\mathbb{Z}^d} c_{k,n} e^{2\pi\imath\beta n\cdot x} \Big) T_{\alpha k}\gamma(x).$$

If $1 < p < \infty$, then, by (11),

$$S_{K,\infty}c = \sum_{|k|\le K} m_k T_{\alpha k}\gamma$$

and, as we have seen in Theorem 1, $S_{K,\infty}c$ converges to $R_g c$ in $W(L_p, \ell_{q,0})(\mathbb{R}^d)$ norm as $K \to \infty$. Gröchenig, Heil, and Okoudjou [17, 18] verified that $S_{K,N}c \to R_g c$ in $W(L_p, \ell_q)(\mathbb{R}^d)$ norm as $K, N \to \infty$ and $1 < p < \infty, 1 \le q < \infty$. Obviously,

$$S_{K,N}c = \sum_{|k|\le K} (S_N m_k) T_{\alpha k}\gamma. \tag{12}$$

If $p = 1$, then the results in (11) are not true. However, using a summability method, say the Fejér's method, we can extend (11). Summability methods are used quite often in Fourier analysis. For the theory of summation see e.g. Butzer and Nessel [2], Trigub and Belinsky [25] and Weisz [28]. The N^{th} Fejér mean of the Fourier series of $h \in L_1(Q_{1/\beta})$ is given by

$$\sigma_N h(x) := \sum_{|n|\le N} \Big(\prod_{j=1}^{d} \Big(1 - \frac{|n_j|}{N+1}\Big)\Big) \hat{h}(n) e^{2\pi\imath\beta n\cdot x} \qquad (N \in \mathbb{N}).$$

Then

$$\sigma_N h \to h \qquad \text{in } L_p(Q_{1/\beta}) \text{ norm and a.e. as } N \to \infty, \tag{13}$$

whenever $1 \le p < \infty$. If h is continuous, then the convergence holds uniformly (see Marcinkiewicz and Zygmund [23, 29] or Weisz [28]).

We define the Fejér means for Gabor series as well: if $c \in s_{p,q}$ $(1 \le p, q \le \infty)$, then let

$$\sigma_{\gamma,K,N}c := \sigma_{K,N}c := \sum_{|k|\le K} \sum_{|n|\le N} \Big(\prod_{j=1}^{d} \Big(1 - \frac{|n_j|}{N+1}\Big)\Big) c_{k,n} M_{\beta n} T_{\alpha k}\gamma.$$

It is easy to see that

$$\sigma_{K,N}c = \sum_{|k|\le K} (\sigma_N m_k) T_{\alpha k}\gamma. \tag{14}$$

Instead of Fejér summation, we may take a general summability method, the so-called θ-summability defined by one single function θ. For $\theta \in W(C, \ell_1)$ the N^{th} θ-mean of the Fourier series of $h \in L_1(Q_{1/\beta})$ resp. of the Gabor series of $c \in s_{p,q}$ $(1 \le p, q \le \infty)$ are defined by

$$\sigma_N^{\theta} h(x) := \sum_{n\in\mathbb{Z}^d} \theta\Big(\frac{-n}{N+1}\Big) \hat{h}(n) e^{2\pi\imath\beta n\cdot x}$$

and

$$\sigma^\theta_{\gamma,K,N}c := \sigma^\theta_{K,N}c := \sum_{|k|\le K}\sum_{n\in\mathbb{Z}^d}\theta\Big(\frac{-n}{N+1}\Big)c_{k,n}M_{\beta n}T_{\alpha k}\gamma \qquad (K,N\in\mathbb{N}).$$

Observe that these series are absolutely convergent because

$$|\hat{h}(n)| \le \|h\|_1, \qquad |c_{k,n}| \le \|m_k\|_1 \le \|c\|_{s_{p,q}}$$

and

$$\sum_{n\in\mathbb{Z}^d}\Big|\theta\Big(\frac{-n}{N+1}\Big)\Big| \le (N+1)^d\|\theta\|_{W(C,\ell_1)} < \infty.$$

We can see immediately that (14) holds in this case, too, namely

$$\sigma^\theta_{K,N}c = \sum_{|k|\le K}(\sigma^\theta_N m_k)T_{\alpha k}\gamma. \tag{15}$$

If $\theta = \mathbf{1}_{[-1,1]^d}$, then we obtain the partial sums; if $\theta(x) = \prod_{j=1}^d \max(0, 1-|x_j|)$, then the Fejér means.

In Feichtinger and Weisz [9, 10] we verified the analogous statements to (13) for θ-summability. If $\hat\theta \in L_1(\mathbb{R}^d)$, then

$$\sigma^\theta_N h \to \theta(0)h \qquad \text{in } L_p(Q_{1/\beta}) \text{ norm as } N\to\infty \tag{16}$$

for all $h \in L_p(Q_{1/\beta})$ $(1\le p<\infty)$. If $h\in C(Q_{1/\beta})$, then the convergence is uniform (see [9]). The almost everywhere convergence is treated in the next section.

Now we are ready to prove the norm convergence of Gabor expansions in amalgam spaces.

Theorem 4 *Assume that $\gamma \in W(L_\infty,\ell_1)(\mathbb{R}^d)$ and $c\in s_{p,q}$.*

(i) If $1<p<\infty$ and $1\le q<\infty$, then

$$\lim_{K,N\to\infty} S_{K,N}c = R_\gamma c \qquad \textit{in } W(L_p,\ell_q)(\mathbb{R}^d) \textit{ norm.}$$

If $q=\infty$, then the convergence holds in the w^* topology of $W(L_p,\ell_\infty)(\mathbb{R}^d)$ and if $c\in s_{p,\infty,0}$, then in $W(L_p,\ell_\infty)(\mathbb{R}^d)$ norm.

(ii) If $\hat\theta\in L_1(\mathbb{R}^d)$, $1\le p<\infty$ and $1\le q<\infty$, then

$$\lim_{K,N\to\infty} \sigma^\theta_{K,N}c = \theta(0)R_\gamma c \qquad \textit{in } W(L_p,\ell_q)(\mathbb{R}^d) \textit{ norm.}$$

If $q=\infty$, then the convergence holds in the w^* topology of $W(L_p,\ell_\infty)(\mathbb{R}^d)$ and if $c\in s_{p,\infty,0}$, then in $W(L_p,\ell_\infty)(\mathbb{R}^d)$ norm.

(iii) If $\hat\theta \in L_1(\mathbb{R}^d)$, $p=\infty$, $1 \le q < \infty$, and m_k is continuous for all $k \in \mathbb{Z}^d$, then

$$\lim_{K,N\to\infty} \sigma^\theta_{K,N} c = \theta(0) R_\gamma c \qquad \text{in } W(L_\infty, \ell_q)(\mathbb{R}^d) \text{ norm.}$$

If $q=\infty$, then the convergence holds in the w^* topology of $L_\infty(\mathbb{R}^d)$ and if $c \in s_{\infty,\infty,0}$, then in $L_\infty(\mathbb{R}^d)$ norm. If in addition γ is continuous as well, then we obtain convergence in $W(C,\ell_q)(\mathbb{R}^d)$ norm if $1 \le q < \infty$ and in $C(\mathbb{R}^d)$ norm if $q=\infty$ and $c \in s_{\infty,\infty,0}$.

Proof. If $c \in s_{p,q,0}$, then for all $\epsilon > 0$ we can find $K_0 = K_0(\epsilon)$ such that

$$\Big(\sum_{|k|>K_0} \|m_k\|^q_{L_p(Q_{1/\beta})} \Big)^{1/q} < \epsilon$$

with the usual modification for $q=\infty$. Using (15) we can write the difference $\theta(0)R_\gamma c - \sigma^\theta_{K,N} c$ in the following form

$$\begin{aligned} \theta(0)R_\gamma c - \sigma^\theta_{K,N} c \;&=\; \Big(\theta(0) R_\gamma c - \theta(0) \sum_{|k|\le K_0} m_k T_{\alpha k}\gamma \Big) \\ &\quad + \Big(\theta(0) \sum_{|k|\le K_0} m_k T_{\alpha k}\gamma - \sum_{|k|\le K_0} (\sigma^\theta_N m_k) T_{\alpha k}\gamma \Big) \\ &\quad + \Big(\sum_{|k|\le K_0} (\sigma^\theta_N m_k) T_{\alpha k}\gamma - \sum_{|k|\le K} (\sigma^\theta_N m_k) T_{\alpha k}\gamma \Big). \end{aligned} \tag{17}$$

Applying Theorem 1, the inequality

$$\|\sigma^\theta_N h\|_{L_p(Q_{1/\beta})} \le C \|h\|_{L_p(Q_{1/\beta})} \qquad (N \in \mathbb{N}^d, 1 \le p \le \infty) \tag{18}$$

and (16) we conclude that

$$\begin{aligned} &\|\theta(0)R_\gamma c - \sigma^\theta_{K,N} c\|_{W(L_p,\ell_{q,0})} \\ &\le\; |\theta(0)| \Big\| \sum_{|k|>K_0} m_k T_{\alpha k}\gamma \Big\|_{W(L_p,\ell_{q,0})} \\ &\quad + \Big\| \sum_{|k|\le K_0} (\theta(0) m_k - \sigma^\theta_N m_k) T_{\alpha k}\gamma \Big\|_{W(L_p,\ell_{q,0})} \\ &\quad + \Big\| \sum_{K_0<|k|\le K} (\sigma^\theta_N m_k) T_{\alpha k}\gamma \Big\|_{W(L_p,\ell_{q,0})} \\ &\le\; C |\theta(0)| \|\gamma\|_{W(L_\infty,\ell_1)} \Big(\sum_{|k|>K_0} \|m_k\|^q_{L_p(Q_{1/\beta})} \Big)^{1/q} \\ &\quad + C \|\gamma\|_{W(L_\infty,\ell_1)} \Big(\sum_{|k|\le K_0} \|\theta(0) m_k - \sigma^\theta_N m_k\|^q_{L_p(Q_{1/\beta})} \Big)^{1/q} \end{aligned}$$

$$
\begin{aligned}
&+ C\|\gamma\|_{W(L_\infty,\ell_1)}\Big(\sum_{K_0<|k|\le K}\|\sigma_N^\theta m_k\|^q_{L_p(Q_{1/\beta})}\Big)^{1/q}\\
\le\quad & C\epsilon\|\gamma\|_{W(L_\infty,\ell_1)},
\end{aligned}
$$

if N is sufficiently large and $K > K_0$, which shows (ii) and (iii) for $1 \le q < \infty$ or $q = \infty$ and $c \in s_{p,\infty,0}$.

To prove the w^* convergence for $q = \infty$, let $h \in W(L_{p'}, \ell_1)(\mathbb{R}^d)$. We get by (17) that

$$
\begin{aligned}
\Big|\Big\langle\theta(0)R_\gamma c - \sigma^\theta_{K,N}c, h\Big\rangle\Big| \quad\le\quad & |\theta(0)|\Big|\Big\langle\sum_{|k|>K_0} m_k T_{\alpha k}\gamma, h\Big\rangle\Big|\\
&+\Big|\Big\langle\sum_{|k|\le K_0}(\theta(0)m_k - \sigma_N^\theta m_k)T_{\alpha k}\gamma, h\Big\rangle\Big|\\
&+\Big|\Big\langle\sum_{K_0<|k|\le K}(\sigma_N^\theta m_k)T_{\alpha k}\gamma, h\Big\rangle\Big|.
\end{aligned}
$$

The first and third term is small if K_0 is sufficiently large, because of (4) and (18). We obtain for the second term analogously to (4) that

$$
\begin{aligned}
&\Big|\Big\langle\sum_{|k|\le K_0}(\theta(0)m_k - \sigma_N^\theta m_k)T_{\alpha k}\gamma, h\Big\rangle\Big|\\
&\quad\le\quad C\sum_{l\in\mathbb{Z}^d}\sum_{|k|\le K_0}\|\gamma T_{\alpha l}\mathbf{1}_{Q_\alpha}\|_\infty\|\theta(0)m_k - \sigma_N^\theta m_k\|_{L_p(Q_{1/\beta})}\|hT_{\alpha k+\alpha l}\mathbf{1}_{Q_\alpha}\|_{p'},
\end{aligned}
$$

and, by (16), this converges to 0 as $N \to \infty$. This proves the w^* convergence in (ii) and (iii). With the help of (11) the statement (i) can be proven similarly. ∎

Note that (i) was proved for $1 \le q < \infty$ by Gröchenig, Heil, and Okoudjou [17, 18]. Let us apply this theorem to $c = C_g f$. The following notations will be used:

$$
S_{g,\gamma,K,N}f := S_{K,N}f := S_{K,N}(C_g f) = \sum_{|k|\le K}\sum_{|n|\le N}\langle f, M_{\beta n}T_{\alpha k}g\rangle M_{\beta n}T_{\alpha k}\gamma,
$$

$$
\begin{aligned}
\sigma^\theta_{g,\gamma,K,N}f := \sigma^\theta_{K,N}f \quad:=\quad & \sigma^\theta_{K,N}(C_g f)\\
=\quad & \sum_{|k|\le K}\sum_{n\in\mathbb{Z}^d}\theta\Big(\frac{-n}{N+1}\Big)\langle f, M_{\beta n}T_{\alpha k}g\rangle M_{\beta n}T_{\alpha k}\gamma.
\end{aligned}
$$

Corollary 3 *Assume that $g, \gamma \in W(L_\infty, \ell_1)(\mathbb{R}^d)$ such that $\mathcal{G}(g, \alpha, \beta)$ is a Gabor frame for $L_2(\mathbb{R}^d)$ with dual frame $\mathcal{G}(\gamma, \alpha, \beta)$. Let $f \in W(L_p, \ell_q)(\mathbb{R}^d)$.*

(i) If $1 < p < \infty$ *and* $1 \le q < \infty$*, then*

$$\lim_{K,N\to\infty} S_{K,N} f = f \qquad \text{in } W(L_p,\ell_q)(\mathbb{R}^d) \text{ norm.}$$

If $q = \infty$*, then the convergence holds in the* w^* *topology of* $W(L_p,\ell_\infty)(\mathbb{R}^d)$ *and if* $f \in W(L_p,\ell_{\infty,0})(\mathbb{R}^d)$*, then in* $W(L_p,\ell_\infty)(\mathbb{R}^d)$ *norm.*

(ii) If $\hat\theta \in L_1(\mathbb{R}^d)$*,* $1 \le p < \infty$ *and* $1 \le q < \infty$*, then*

$$\lim_{K,N\to\infty} \sigma^\theta_{K,N} f = \theta(0) f \qquad \text{in } W(L_p,\ell_q)(\mathbb{R}^d) \text{ norm.}$$

If $q = \infty$*, then the convergence holds in the* w^* *topology of* $W(L_p,\ell_\infty)(\mathbb{R}^d)$ *and if* $f \in W(L_p,\ell_{\infty,0})(\mathbb{R}^d)$*, then in* $W(L_p,\ell_\infty)(\mathbb{R}^d)$ *norm.*

(iii) If $\hat\theta \in L_1(\mathbb{R}^d)$*,* $p = \infty$*,* $1 \le q < \infty$ *and* f *and* g *are continuous then*

$$\lim_{K,N\to\infty} \sigma^\theta_{K,N} f = \theta(0) f \qquad \text{in } W(L_\infty,\ell_q)(\mathbb{R}^d) \text{ norm.}$$

If $q = \infty$*, then the convergence holds in the* w^* *topology of* $L_\infty(\mathbb{R}^d)$ *and if* $f \in L_{\infty,0}(\mathbb{R}^d)$*, then in* $L_\infty(\mathbb{R}^d)$ *norm. If in addition* γ *is continuous as well, then we get convergence in* $W(C,\ell_q)(\mathbb{R}^d)$ *norm if* $1 \le q < \infty$ *and in* $C(\mathbb{R}^d)$ *norm if* $q = \infty$ *and* $f \in C_0(\mathbb{R}^d)$*.*

Proof. This corollary follows from Theorem 4 and Corollary 2. By (7), if $p = \infty$ and f and g are continuous, then the functions m_k $(k \in \mathbb{Z}^d)$ are continuous, too. ■

Note that Fejér summation of Gabor series for L_p spaces was also investigated in Grafakos and Lennard [15] and Lyubarskii and Seip [22].

All the results of this section can also be proven for rectangular partial sums. Namely, if we define $S_{K,N}c$ and $\sigma^\theta_{K,N}c$ by

$$S_{K,N}c := \sum_{k_1=-K_1}^{K_1} \cdots \sum_{k_d=-K_d}^{K_d} \sum_{n_1=-N_1}^{N_1} \cdots \sum_{n_d=-N_d}^{N_d} c_{k,n} M_{\beta n} T_{\alpha k}\gamma$$

and

$$\sigma^\theta_{K,N}c := \sum_{k_1=-K_1}^{K_1} \cdots \sum_{k_d=-K_d}^{K_d} \sum_{n\in\mathbb{Z}^d} \theta\Big(\frac{-n_1}{N_1+1},\ldots,\frac{-n_d}{N_d+1}\Big) c_{k,n} M_{\beta n} T_{\alpha k}\gamma,$$

$(K,N \in \mathbb{N}^d)$ then the same theorems hold. In this case under $K,N\to\infty$ we mean that $K_j,N_j\to\infty$ for all $j=1,\ldots,d$.

5 A.e. convergence of Gabor expansions

First, we investigate the a.e. convergence of summations of Fourier series. In Feichtinger and Weisz [10] we applied the homogeneous Herz spaces in summability theory. $\dot{E}_q(\mathbb{R}^d)$ contains all measurable functions f for which

$$\|f\|_{\dot{E}_q} := \sum_{k=-\infty}^{\infty} 2^{kd(1-1/q)} \|f \mathbf{1}_{P_k}\|_q < \infty,$$

where $P_k := \{x : |x| < 2^k\} \setminus \{x : |x| \geq 2^{k-1}\}$. These spaces are special cases of the Herz spaces [19] (see also Feichtinger [8], Garcia-Cuerva and Herrero [13]). It is easy to see that

$$L_1(\mathbb{R}^d) = \dot{E}_1(\mathbb{R}^d) \hookleftarrow \dot{E}_q(\mathbb{R}^d) \hookleftarrow \dot{E}_r(\mathbb{R}^d) \hookleftarrow \dot{E}_\infty(\mathbb{R}^d), \qquad 1 < q < r < \infty.$$

In this way we obtained ([10]) the following result: if $\hat{\theta} \in \dot{E}_{p'}(\mathbb{R}^d)$, then

$$\sigma_N^\theta h \to \theta(0) h \qquad \text{a.e. as } N \to \infty \tag{19}$$

for all $h \in L_p(Q_{1/\beta})$, where $1 \leq p < \infty$ and $1/p + 1/p' = 1$. Actually, the convergence holds at every Lebesgue point. Some sufficient conditions for θ such that $\hat{\theta} \in \dot{E}_{r'}(\mathbb{R}^d)$ and many examples can be found in [10].

These results are generalized for Gabor series as follows.

Theorem 5 *Assume that $\gamma \in L_\infty(\mathbb{R}^d)$ with compact support and $c \in s_{p,q}$.*

(i) If $1 < p < \infty$ and $1 \leq q \leq \infty$, then

$$\lim_{K,N\to\infty} S_{K,N} c = R_\gamma c \qquad a.e.$$

(ii) If $\hat{\theta} \in \dot{E}_{r'}(\mathbb{R}^d)$, $1 \leq r \leq p < \infty$, $1/r + 1/r' = 1$ and $1 \leq q \leq \infty$, then

$$\lim_{K,N\to\infty} \sigma_{K,N}^\theta c = \theta(0) R_\gamma c \qquad a.e.$$

Proof. Taking (17) for a fixed x, we observe that the first and third term on the right hand side is equal to 0, if K_0 is large enough, since γ has compact support. Thus,

$$\begin{aligned} |\theta(0)R_\gamma c(x) - \sigma_{K,N}^\theta c(x)| &= \Big| \sum_{|k|\leq K_0} (\theta(0) m_k(x) - \sigma_N^\theta m_k(x)) T_{\alpha k}\gamma(x) \Big| \\ &\leq \|\gamma\|_\infty \sum_{|k|\leq K_0} |\theta(0) m_k(x) - \sigma_N^\theta m_k(x)| \end{aligned}$$

and (19) proves the theorem. ∎

Note that $s_{p_1,q} \hookleftarrow s_{p_2,q}$ if $p_1 \leq p_2$ and $s_{p,q_1} \hookrightarrow s_{p,q_2}$ if $q_1 \leq q_2$.

In order to extend this theorem to functions $\gamma \in W(L_\infty, \ell_1)(\mathbb{R}^d)$, which lack compact support, we introduce the *maximal operators* S_* and σ_*^θ by

$$S_{\gamma,*}c := S_*c := \sup_{K,N\in\mathbb{N}} |S_{K,N}c|, \quad S_{g,\gamma,*}f := S_*f := S_*(C_g f) := \sup_{K,N\in\mathbb{N}} |S_{K,N}f|,$$

$$\sigma_{\gamma,*}^\theta c := \sigma_*^\theta c := \sup_{K,N\in\mathbb{N}} |\sigma_{K,N}^\theta c|, \quad \sigma_{g,\gamma,*}^\theta := \sigma_*^\theta f := \sigma_*^\theta(C_g f) := \sup_{K,N\in\mathbb{N}} |\sigma_{K,N}^\theta f|.$$

For the trigonometric Fourier series of $h \in L_p(Q_{1/\beta})$ we use analogous notations. It is known that

$$\|S_*h\|_{L_p(Q_{1/\beta})} \leq C_p \|h\|_{L_p(Q_{1/\beta})} \qquad (1 < p < \infty) \tag{20}$$

and

$$\|\sigma_*^\theta h\|_{L_p(Q_{1/\beta})} \leq C_p \|h\|_{L_p(Q_{1/\beta})} \qquad (1 \leq r < p < \infty), \tag{21}$$

whenever $\hat{\theta} \in \dot{E}_{r'}$ (see Carleson [3], Hunt [20], Fefferman [7], Grafakos [14] and Feichtinger and Weisz [10]). Now we prove similar inequalities for Gabor series.

Theorem 6 *Assume that $g, \gamma \in W(L_\infty, \ell_1)(\mathbb{R}^d)$, $c \in s_{p,q}$ and $f \in W(L_p, \ell_q)(\mathbb{R}^d)$.*

(i) If $1 < p < \infty$ and $1 \leq q \leq \infty$, then

$$\|S_*c\|_{W(L_p,\ell_q)} \leq C_p \|\gamma\|_{W(L_\infty,\ell_1)} \|c\|_{s_{p,q}}, \tag{22}$$

$$\|S_*f\|_{W(L_p,\ell_q)} \leq C_p \|g\|_{W(L_\infty,\ell_1)} \|\gamma\|_{W(L_\infty,\ell_1)} \|f\|_{W(L_p,\ell_q)}. \tag{23}$$

(ii) If $\hat{\theta} \in \dot{E}_{r'}(\mathbb{R}^d)$, $1 \leq r < p < \infty$, $1/r + 1/r' = 1$ and $1 \leq q \leq \infty$, then

$$\|\sigma_*^\theta c\|_{W(L_p,\ell_q)} \leq C_p \|\gamma\|_{W(L_\infty,\ell_1)} \|c\|_{s_{p,q}}, \tag{24}$$

$$\|\sigma_*^\theta f\|_{W(L_p,\ell_q)} \leq C_p \|g\|_{W(L_\infty,\ell_1)} \|\gamma\|_{W(L_\infty,\ell_1)} \|f\|_{W(L_p,\ell_q)}. \tag{25}$$

Proof. By (15),

$$\sigma_*^\theta c \leq \sum_{k\in\mathbb{Z}^d} (\sigma_*^\theta m_k) |T_{\alpha k}\gamma|.$$

Using Theorem (1) and (21), we obtain

$$\begin{aligned}
\|\sigma_*^\theta c\|_{W(L_p,\ell_q)} &\leq C\|\gamma\|_{W(L_\infty,\ell_1)} \Big(\sum_{k\in\mathbb{Z}^d} \|\sigma_*^\theta m_k\|_{L_p(Q_{1/\beta})}^q \Big)^{1/q} \\
&\leq C_p \|\gamma\|_{W(L_\infty,\ell_1)} \Big(\sum_{k\in\mathbb{Z}^d} \|m_k\|_{L_p(Q_{1/\beta})}^q \Big)^{1/q} \\
&= C_p \|\gamma\|_{W(L_\infty,\ell_1)} \|c\|_{s_{p,q}},
\end{aligned}$$

which proves (24). (25) comes from Theorem 2. The inequalities for S_* can be shown similarly. ∎

By (21) we can see as in (5) that $\sigma_*^\theta m_k \to 0$ a.e. as $k \to \infty$, whenever $c \in s_{p,q}$, $\hat{\theta} \in \dot{E}_{r'}(\mathbb{R}^d)$, $1 \le r < p < \infty$ and $q \le p$. Using (17) we could verify Theorem 5 for a general $\gamma \in W(L_\infty, \ell_1)(\mathbb{R}^d)$ and for $1 \le r < p < \infty$, $q \le p$. However, the next result is more general.

Theorem 7 *Assume that $\gamma \in W(L_\infty, \ell_1)(\mathbb{R}^d)$ and $c \in s_{p,q}$.*

(i) If $1 < p < \infty$ and $1 \le q \le \infty$, then

$$\lim_{K,N\to\infty} S_{K,N}c = R_\gamma c \qquad a.e.$$

(ii) If $\hat{\theta} \in \dot{E}_{r'}(\mathbb{R}^d)$, $1 \le r < p < \infty$, $1/r + 1/r' = 1$ and $1 \le q \le \infty$, then

$$\lim_{K,N\to\infty} \sigma_{K,N}^\theta c = \theta(0) R_\gamma c \qquad a.e.$$

Proof. Fix $c \in s_{p,q}$ and set

$$\xi := \limsup_{K,N\to\infty} |\sigma_{\gamma,K,N}^\theta c - \theta(0) R_\gamma c|.$$

For (ii) it is sufficient to show that $\xi = 0$ a.e.

Choose $\gamma_m \in W(L_\infty, \ell_1)(\mathbb{R}^d)$ with compact support such that

$$\|\gamma - \gamma_m\|_{W(L_\infty,\ell_1)} \to 0 \quad \text{as} \quad m \to \infty.$$

By Theorem 5,

$$\begin{aligned}
\xi &\le \limsup_{K,N\to\infty} |\sigma_{\gamma,K,N}^\theta c - \sigma_{\gamma_m,K,N}^\theta c| \\
&\quad + \limsup_{K,N\to\infty} |\sigma_{\gamma_m,K,N}^\theta c - \theta(0) R_{\gamma_m} c| + |\theta(0) R_{\gamma_m} c - \theta(0) R_\gamma c| \\
&\le \sigma_{\gamma-\gamma_m,*}^\theta c + |\theta(0) R_{\gamma-\gamma_m} c|
\end{aligned}$$

for all $m \in \mathbb{N}$. Taking into account Theorems 1 and 6, we conclude

$$\begin{aligned}
\|\xi\|_{W(L_p,\ell_q)} &\le \|\sigma_{\gamma-\gamma_m,*}^\theta c\|_{W(L_p,\ell_q)} + \|\theta(0) R_{\gamma-\gamma_m} c\|_{W(L_p,\ell_q)} \\
&\le C_p \|\gamma - \gamma_m\|_{W(L_\infty,\ell_1)} \|c\|_{s_{p,q}}
\end{aligned}$$

for all $m \in \mathbb{N}$. Since $\gamma_m \to \gamma$ in $W(L_\infty, \ell_1)(\mathbb{R}^d)$ norm as $m \to \infty$, $\|\xi\|_{W(L_p,\ell_q)} = 0$ and so $\xi = 0$ a.e. (i) can be shown in an analogous way. ■

Corollary 4 *Assume that $g, \gamma \in W(L_\infty, \ell_1)(\mathbb{R}^d)$ such that $\mathcal{G}(g, \alpha, \beta)$ is a Gabor frame for $L_2(\mathbb{R}^d)$ with dual frame $\mathcal{G}(\gamma, \alpha, \beta)$. Let $f \in W(L_p, \ell_q)(\mathbb{R}^d)$.*

(i) If $1 < p < \infty$ and $1 \leq q \leq \infty$, then

$$\lim_{K,N\to\infty} S_{K,N} f = f \qquad a.e.$$

(ii) If $\hat\theta \in \dot E_{r'}(\mathbb{R}^d)$, $1 \leq r < p < \infty$, $1/r + 1/r' = 1$ and $1 \leq q \leq \infty$, then

$$\lim_{K,N\to\infty} \sigma^\theta_{K,N} f = \theta(0) f \qquad a.e.$$

If γ has compact support, then this convergence holds for $1 \leq r \leq p < \infty$.

6 Hardy-Littlewood inequalities

If $p = 2$, then by Parseval formula

$$\|c\|_{s_{2,q}} = \Big(\sum_{k\in\mathbb{Z}^d} \Big(\sum_{n\in\mathbb{Z}^d} |c_{k,n}|^2 \Big)^{q/2} \Big)^{1/q}.$$

Now Theorem 2 implies

$$\Big(\sum_{k\in\mathbb{Z}^d} \Big(\sum_{n\in\mathbb{Z}^d} |\langle f, M_{\beta n} T_{\alpha k} g\rangle|^2 \Big)^{q/2} \Big)^{1/q} = \|C_g f\|_{s_{2,q}} \leq C \|g\|_{W(L_\infty,\ell_1)} \|f\|_{W(L_2,\ell_q)}$$

with the obvious modification for $q = \infty$. Of course, if q also equals 2, then $s_{2,2} = \ell_2$ and $W(L_2, \ell_2) = L_2$. Similarly,

$$\|R_\gamma c\|_{W(L_2,\ell_q)} \leq C \|\gamma\|_{W(L_\infty,\ell_1)} \Big(\sum_{k\in\mathbb{Z}^d} \Big(\sum_{n\in\mathbb{Z}^d} |c_{k,n}|^2 \Big)^{q/2} \Big)^{1/q}.$$

We will generalize these inequalities for $1 < p < \infty$ below.

For Fourier series of $h \in L_p(Q_{1/\beta})$ it is known that

$$\Big(\sum_{n\in\mathbb{Z}^d} \frac{|\hat h(n)|^p}{((|n_1|+1)\cdots(|n_d|+1))^{2-p}} \Big)^{1/p} \leq C_p \|h\|_{L_p(Q_{1/\beta})} \qquad (1 < p \leq 2)$$

and

$$\|h\|_{L_p(Q_{1/\beta})} \leq C_p \Big(\sum_{n\in\mathbb{Z}^d} \frac{|\hat h(n)|^p}{((|n_1|+1)\cdots(|n_d|+1))^{2-p}} \Big)^{1/p} \qquad (2 \leq p < \infty)$$

(see Edwards [5], Jawerth and Torchinsky [21] and Weisz [27]).

Theorem 8 *Assume that $g \in W(L_\infty, \ell_1)(\mathbb{R}^d)$ and $f \in W(L_p, \ell_q)(\mathbb{R}^d)$ for some $1 < p \leq 2$, $1 \leq q \leq \infty$. Then*

$$\Big(\sum_{k\in\mathbb{Z}^d} \Big(\sum_{n\in\mathbb{Z}^d} \frac{|\langle f, M_{\beta n} T_{\alpha k} g\rangle|^p}{((|n_1|+1)\cdots(|n_d|+1))^{2-p}} \Big)^{q/p} \Big)^{1/q} \leq C_p \|g\|_{W(L_\infty,\ell_1)} \|f\|_{W(L_p,\ell_q)}.$$

Proof. The proof follows from

$$\begin{aligned}\Big(\sum_{k\in\mathbb{Z}^d}\Big(\sum_{n\in\mathbb{Z}^d}\frac{|\langle f,M_{\beta n}T_{\alpha k}g\rangle|^p}{((|n_1|+1)\cdots(|n_d|+1))^{2-p}}\Big)^{q/p}\Big)^{1/q} & \\ &\leq C_p\Big(\sum_{k\in\mathbb{Z}^d}\|m_k\|^q_{L_p(Q_{1/\beta})}\Big)^{1/q}\\ &= C_p\|C_g f\|_{s_{p,q}}\end{aligned}$$

and from Theorem 2. ∎

We obtain the next theorem in the same way.

Theorem 9 *Assume that* $\gamma\in W(L_\infty,\ell_1)(\mathbb{R}^d)$ *and*

$$\Big(\sum_{k\in\mathbb{Z}^d}\Big(\sum_{n\in\mathbb{Z}^d}\frac{|c_{k,n}|^p}{((|n_1|+1)\cdots(|n_d|+1))^{2-p}}\Big)^{q/p}\Big)^{1/q}$$

is finite for some $2\leq p<\infty$ *and* $1\leq q\leq\infty$. *Then* $R_\gamma c\in W(L_p,\ell_q)$ *and*

$$\begin{aligned}&\|R_\gamma c\|_{W(L_p,\ell_q)}\\ &\leq\quad C_p\|\gamma\|_{W(L_\infty,\ell_1)}\Big(\sum_{k\in\mathbb{Z}^d}\Big(\sum_{n\in\mathbb{Z}^d}\frac{|c_{k,n}|^p}{((|n_1|+1)\cdots(|n_d|+1))^{2-p}}\Big)^{q/p}\Big)^{1/q}.\end{aligned}$$

7 Marcinkiewicz multiplier theorem

To avoid some technical difficulties, the theorem will be formulated for the one-dimensional case only. However, it can be simply generalized for higher dimensions.

For a given *multiplier* $\lambda=(\lambda_n,n\in\mathbb{Z})$ where the λ_j's are complex numbers, the *multiplier operator* is defined for Fourier series by

$$M_\lambda h(x):=\sum_{n\in\mathbb{Z}}\lambda_n\hat{h}(n)e^{2\pi\imath\beta n\cdot x},$$

where $h\in L_p(Q_{1/\beta})$ $(1<p<\infty)$.

The Marcinkiewicz multiplier theorem says that if

$$|\lambda_i|\leq C,\qquad\sum_{|n|=2^i}^{2^{i+1}-1}|\lambda_n-\lambda_{n+1}|\leq C\qquad(i\in\mathbb{N}),\tag{26}$$

then $M_\lambda h\in L_p(Q_{1/\beta})$ and

$$\|M_\lambda h\|_{L_p(Q_{1/\beta})}\leq C_p\|h\|_{L_p(Q_{1/\beta})}\qquad(1<p<\infty)$$

(see Zygmund [29, Vol. II. p. 232], and for the multi-dimensional case Edwards and Gaudry [6] and Weisz [26]).

For Gabor series let formally

$$M_\lambda f = \sum_{k,n\in\mathbb{Z}} \lambda_n \langle f, M_{\beta n}T_{\alpha k}g\rangle M_{\beta n}T_{\alpha k}\gamma.$$

As done before, we take the sum first in n:

$$M_\lambda f(x) := \sum_{k\in\mathbb{Z}} \Big(\sum_{n\in\mathbb{Z}} \lambda_n \langle f, M_{\beta n}T_{\alpha k}g\rangle e^{2\pi\imath\beta n\cdot x}\Big) T_{\alpha k}\gamma(x).$$

It is easy to see that the operator M_λ is well defined for $f \in W(L_p,\ell_q)(\mathbb{R})$, $1<p<\infty$, $1\le q\le\infty$.

Theorem 10 *Assume that $g,\gamma \in W(L_\infty,\ell_1)(\mathbb{R})$ and $f\in W(L_p,\ell_q)(\mathbb{R})$ for some $1<p<\infty$ and $1\le q\le\infty$. If (26) holds, then $M_\lambda f \in W(L_p,\ell_q)(\mathbb{R})$ and*

$$\|M_\lambda f\|_{W(L_p,\ell_q)} \le C_p \|\gamma\|_{W(L_\infty,\ell_1)}\|g\|_{W(L_\infty,\ell_1)}\|f\|_{W(L_p,\ell_q)}.$$

Proof. It is easy to see that

$$M_\lambda f = R_\gamma\Big((\lambda_n C_g f(k,n))_{k,n\in\mathbb{Z}}\Big).$$

Then

$$\begin{aligned}
\|M_\lambda f\|_{W(L_p,\ell_q)} &\le C\|\gamma\|_{W(L_\infty,\ell_1)}\Big(\sum_{k\in\mathbb{Z}} \|M_\lambda m_k\|^q_{L_p(Q_{1/\beta})}\Big)^{1/q} \\
&\le C_p\|\gamma\|_{W(L_\infty,\ell_1)}\Big(\sum_{k\in\mathbb{Z}} \|m_k\|^q_{L_p(Q_{1/\beta})}\Big)^{1/q} \\
&\le C_p\|\gamma\|_{W(L_\infty,\ell_1)}\|g\|_{W(L_\infty,\ell_1)}\|f\|_{W(L_p,\ell_q)},
\end{aligned}$$

which finishes the proof of the theorem. ∎

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