SAMPLING THEORY IN SIGNAL AND IMAGE PROCESSING Vol. 6, No. 2, May 2007, pp. 151-166
ISSN: 1530-6429

Banach Framed, Decay in the Context of Localization

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Abstract

We introduce a new definition of localization for frames which eliminates the dependence on the indexing of the frames. Two main results of Gröchenig are extended to this definition, namely that the dual of a localized frame is also localized, and a frame localized with respect to another frame is a Banach frame for the associated family of Banach spaces. These results parallel the results of a more recent paper by Fornasier and Gröchenig.

Key words and phrases : frame, localized frame, Banach frame, Modulation spaces

2000 AMS Mathematics Subject Classification — 42C15, 46E99, 46B15, 47B37, 41A63

1 Introduction

The concept of localization of frames was recently introduced independently by Gröchenig [20] and the group consisting of Balan, Casazza, Heil, and Landau (BCHL) [4]. To understand localization of frames, let $\mathcal{H} = L^2(\mathbb{R})$ and consider two frames $\mathcal{F} = (f_x)_{x\in X}$ and $\mathcal{E} = (e_y)_{y\in Y}$ of $\mathcal{H}$ whose indexing sets X and Y are countable subsets of $\mathbb{R}$. We can think of each $f_x \in \mathcal{F}$ as being "concentrated" near x, and similarly for each $e_y \in \mathcal{E}$. Roughly, $\mathcal{F}$ *is localized with respect to* $\mathcal{E}$ if each f_x can be well-approximated by a finite linear combination of e_y's. In other words, since f_x can be written as

$$f_x = \sum_{y\in Y} \langle f_x, e_y\rangle \tilde{e}_y$$

we say that $\mathcal{F}$ is localized with respect to $\mathcal{E}$ if the magnitudes of the coefficients $|\langle f_x, e_y\rangle|$ exhibit a certain decay as the distance between x and y increases. Equivalently, $\mathcal{F}$ is localized with respect to $\mathcal{E}$ if the cross-Gramian matrix $[\langle f_x, e_y\rangle]_{x\in X, y\in Y}$ has a decay of the diagonal.

Localization has already proven to be a powerful new quality of frames [19, 20, 9, 4, 5, 6, 14, 15, 21]. Gröchenig proved that a frame localized with respect to a Riesz basis is automatically a Banach frame for an often important family of Banach spaces associated with the Riesz basis. This result generalized results from sampling theory, time-frequency analysis, and wavelet analysis [17, 1, 12, 9]. Later, Fornasier and Gröchenig [14] proved a similar result for frames localized with respect to other frames, where the localization is defined by the Gramian matrices belonging to a particular spectral algebra. The role of Banach matrix algebras in this theory of localized frames was further developed by Gröchenig and Leinert [21]. Most recently, Fornasier and Rauhut [15] introduced the notion of continuous localized frames indexed by a locally compact space endowed with a Radon measure and showed that these frames can be sampled to create discrete localized frames. Independent of these developments, BCHL introduced the notion of localized frames to prove powerful results concerning the density and excess of frames, extending their results in [2] and [3]. They also provided an illuminating new perspective on previously known results concerning Gabor frames.

In this paper, we introduce a definition which eliminates the dependence of localization on the indexing of the frames. The property of localization was dependent on indexing for all previous definitions. We then show that the main results of Gröchenig still hold. This definition also extends the results of BCHL, which can be found in [16]. It should be noted that this definition was developed prior to knowledge of the following papers, [14, 15, 21]. The Fornasier and Rauhut definition of localization [15] may be thought of as the most general definition, as it is for continuous frames indexed on a locally compact space with Radon measure. However, the localization condition depends on the prescribed choice of Radon measure, which corresponds to a particular choice of arrangement of the index set. Additionally, in the papers just mentioned, the localization condition is generalized so that the cross-Gramian matrices of localized frames belong to a particular Banach algebra. In this paper we only treat cross-Gramian matrices with off-diagonal ℓ^p decay. Note that matrices with off-diagonal ℓ^1 decay can be thought of as belonging to the Sjöstrand algebra. We expect that this definition can be extended using more general Banach algebras.

Our paper will be organized as follows. In Section 2 we introduce the symmetric definition of localization. In Section 3 we investigate the equivalence structure of ℓ^1-self-localized frames and, finally, in Section 4 we extend the two main results of Gröchenig, namely localization of the dual frame and the construction of Banach frames.

2 Symmetric Localization

Before introducing the new definition, we fix basic notation. We recommend [8], [18], [7], [11], and [10] for additional background.

For a countable set X, let $\mathcal{F} = (f_x)_{x\in X}$ be a **frame** for a separable Hilbert space $\mathcal{H}$ with frame bounds A, B. The **analysis operator** will be denoted $C := C_{\mathcal{F}} : \mathcal{H} \to \ell^2(X)$, where $C(f) = (\langle f, f_x\rangle)_{x\in X}$. The **synthesis operator** will be denoted $D := D_{\mathcal{F}} : \ell^2(X) \to \mathcal{H}$, where $D((c_x)_{x\in X}) = \sum_{x\in X} c_x f_x$. D is the adjoint of C, $D = C^*$. The **frame operator** denoted $S = DC : \mathcal{H} \to \mathcal{H}$, $Sf = \sum_{x\in X} \langle f, f_x\rangle f_x$ is a positive, invertible operator such that $A\cdot I \leq S \leq B\cdot I$. The **canonical dual frame** of $\mathcal{F}$ is denoted $\tilde{\mathcal{F}} = (S^{-1}f_x)_{x\in X} = (\tilde{f}_x)_{x\in X}$ and is such that $f = \sum_{x\in X} \langle f, \tilde{f}_x\rangle f_x = \sum_{x\in X} \langle f, \tilde{f}_x\rangle f_x$ for all $f \in \mathcal{H}$.

In the following, let G be a group of the form $\displaystyle\prod_{i=1}^{d} a_i\mathbb{Z} \times \prod_{j=1}^{e} \mathbb{Z}_{b_j}$. For every $g = (a_1n_1, a_2n_2, ..., a_dn_d, m_1, m_2, ..., m_e) \in G$, let

$$|g| = \sup\{|a_1n_1|, |a_2n_2|, ..., |a_dn_d|, \delta(m_1), \delta(m_2), ..., \delta(m_e)\}$$

$$\text{where } \delta(m) = \begin{cases} 0 & \text{if m=0;} \\ 1 & \text{otherwise.} \end{cases}$$

Define a metric on G to be $d(g, h) = |g - h|$ for $g, h \in G$.

Let $S_n(j)$ be defined to be the ball of radius n centered at j in G. We define $|S_n(j)| := \#[S_n(j)]$, the cardinality of $S_n(j)$.

Definition 2.1 (Symmetric localization). *Let sequences $\mathcal{F} = (f_x)_{x\in X}$ and $\mathcal{E} = (e_y)_{y\in Y}$ in a Hilbert space $\mathcal{H}$, X and Y arbitrary index sets.*

*(1) $(\mathcal{F}, \mathcal{E})$ is **symmetrically ℓ^p-localized** if there exist maps $a_X : X \to G$, $a_Y : Y \to G$ such that $\max\{\sup_{j\in G} |a_X^{-1}(j)|, \sup_{j\in G} |a_Y^{-1}(j)|\} \leq K < \infty$, and $r \in \ell^p(G)$ such that for all $x \in X, y \in Y$,*

$$|\langle f_x, e_y\rangle| \leq r_{a_X(x)-a_Y(y)}.$$

*(2) $\mathcal{F}$ is **symmetrically ℓ^p-self-localized** if it is symmetrically ℓ^p-localized with respect to itself.*

*(3) $(\mathcal{F}, \mathcal{E})$ has **uniform ℓ^p column decay** if for every $\epsilon > 0$ there is a $N_\epsilon > 0$ such that for all $y \in Y$,*

$$\sum_{x\in X\setminus a_X^{-1}(S_{N_\epsilon}(a_Y(y)))} |\langle f_x, e_y\rangle|^p < \epsilon.$$

(4) $(\mathcal{F}, \mathcal{E})$ *has* ***uniform*** ℓ^p ***row decay*** *if for every* $\epsilon > 0$ *there is a* $N_\epsilon > 0$ *such that for all* $x \in X$,

$$\sum_{y \in Y \setminus a_Y^{-1}(S_{N_\epsilon}(a_X(x)))} |\langle f_x, e_y \rangle|^p < \epsilon.$$

Remark 2.2. The terms column and row decay come from considering the cross-Gramian matrix $(\langle f_x, e_y \rangle)_{x \in X, y \in Y}$.

Remark 2.3. If we let $Y = G$ and $a_Y = id$, then we have the definition of BCHL provided $\sup_{j \in G} |a_X^{-1}(j)| \leq K < \infty$. Bounded point inverses are not only desired in applications but also used in nearly all of the theorems of BCHL, so it is not a restrictive condition.

Though we do not have a straight generalization of Gröchenig's definition, every frame localized with respect to a Riesz basis in Gröchenig's sense is localized in this symmetric sense as will be made clear below.

Definition 2.4. *A set* $X \subseteq \mathbb{R}^d$ *is* ***separated*** *if there exists a positive constant* c *such that for every* $x, y \in X$ *such that* $x \neq y$, $0 < c \leq |x - y|$.

Definition 2.5 (Gröchenig [20]). *The frame* $\mathcal{F} = (f_x)_{x \in X}$ *in* $L^2(\mathbb{R}^d)$ *is* ***polynomially localized*** *with respect to the Riesz basis* $\mathcal{E} = (e_y)_{y \in Y}$ *in* $L^2(\mathbb{R}^d)$, *with decay* $s > 0$ *(or* ***s-localized****), where* X *is a finite union of separated sets of* $\mathbb{R}^d$ *and* Y *is a separated set of* $\mathbb{R}^d$, *if for all* $x \in X, y \in Y$, *and* $C > 0$,

$$|\langle f_x, e_y \rangle| \leq C(1 + |x - y|)^{-s} \text{ and } |\langle f_x, \tilde{e}_y \rangle| \leq C(1 + |x - y|)^{-s}.$$

Likewise, $\mathcal{F}$ *is called* ***exponentially localized with exponent*** $\alpha > 0$ *if for some* $\alpha > 0$ *and* $C > 0$,

$$|\langle f_x, e_y \rangle| \leq Ce^{-\alpha|x-y|} \text{ and } |\langle f_x, \tilde{e}_y \rangle| \leq Ce^{-\alpha|x-y|}.$$

To prove that every polynomially localized frame is symmetrically localized, let $\mathcal{F} = (f_x)_{x \in X}$ be s-localized with respect to a Riesz basis $\mathcal{E} = (e_y)_{y \in Y}$ as above. Let $G := \frac{1}{2d}\mathbb{Z}^d$. For $x = (x_i)_{i=1}^d \in \mathbb{R}^d$, let $n_i \leq x_i < n_i + \frac{1}{2d}$ with $n_i \in \frac{1}{2d}\mathbb{Z}, i = 1, \ldots d$. We define $a_X : X \to G$ in the following way:

$$a_X(x) = (w_1, \ldots, w_d) \text{ where } w_i = \begin{cases} n_i & \text{if } x_i < n_i + \frac{1}{4d} \\ n_i + \frac{1}{2d} & \text{if } x_i \geq n_i + \frac{1}{4d}. \end{cases}$$

We define $a_Y : Y \to G$ similarly.

If we define $r : G \to \mathbb{C}$ to be $r = (r_g)_{g \in G} = (C(\frac{1}{2} + |g|)^{-s})_{g \in G}$, then $r \in \ell^p$ where $p > \frac{1}{s}$, and $\mathcal{F}$ is symmetrically ℓ^p-localized with respect to $\mathcal{E}$ with decay given by r. A similar argument holds for exponentially localized frames. □

The most compelling reason for the introduction of this new definition for localized frames is the need for a definition that not only compares any two frames, but also comapres two frames regardless of the indexing. Recall that the frame series converges unconditionally or equivalently; the convergence of the partial sums is independent of the indexing of the frames. All previous definitions depend on the indexing. Given a frame $\mathcal{F}$ localized with respect to a Riesz basis $\mathcal{E}$ in the sense of Gröchenig [20], we can easily permute the index set of $\mathcal{F}$, $\mathcal{E}$, or both so that $\mathcal{F}$ is no longer localized with respect to $\mathcal{E}$. The same is true for Fornasier and Gröchenig's definition [14], Fornasier and Rauhut's definition [15], and Gröchenig and Leinert's definition [21]. The definition found in [4] is also dependent on the indexing of the frames because of the dependence on the map $a_X : X \to G$. The symmetrically localized definition is independent of the indexing. For this definition, we need only the existence of some maps $a_X : X \to G$ and $a_Y : Y \to G$ with finite point inverses, so given a permutation p of the index set X, we can define a new map $a'_X = a_X \circ p^{-1} : p(X) \to G$.

Example 2.6. Let $\mathcal{F} = \left(\frac{\sin[\pi(x+k)]}{\pi(x+k)}\right)_{k\in\frac{1}{2}\mathbb{Z}}$ and $\mathcal{E} = \left(\frac{\sin[\pi(x+n)]}{\pi(x+n)}\right)_{n\in\mathbb{Z}}$ be frames for $L^2(\mathbb{R})$. Let $a_{\mathbb{Z}} : \mathbb{Z} \to \mathbb{Z}$ be the identity function, and $a_{\frac{1}{2}\mathbb{Z}} : \frac{1}{2}\mathbb{Z} \to \mathbb{Z}$ be defined

$$a_{\frac{1}{2}\mathbb{Z}}(k) = \begin{cases} k & \text{if } k \in \mathbb{Z}; \\ k - 1/2 & \text{if } k \in \mathbb{Z}^+; \\ k + 1/2 & \text{if } k \in \mathbb{Z}^-. \end{cases}$$

$\mathcal{F}$ is symmetrically ℓ^p localized with respect to $\mathcal{E}$ for any $p > 1$, where the decay is given by $r_g = \begin{cases} 1 & \text{if } g = 0; \\ \frac{1}{|g|\pi} & \text{if } g \neq 0. \end{cases}$

Example 2.7. Let $\mathcal{G}(\gamma, \Gamma) = \{T_x M_w \gamma : (x, w) \in \Gamma\}$ and $\mathcal{G}(\gamma, \Lambda) = \{T_y M_z \gamma : (z, y) \in \Lambda\}$ be Gabor frames in $L^2(\mathbb{R})$, where T_x is the translation operator defined $T_x f(y) = f(y-x)$ and M_z is the modulation operator defined $M_z f(y) = e^{2\pi i z y} f(y)$. Suppose $\gamma = e^{-\pi x^2}$, the normalized Gaussian, and $\Gamma, \Lambda \subset \mathbb{R}^2$ with finite multiplicities. Let $a_\Gamma : \Gamma \to \mathbb{Z}\times\mathbb{Z}$ and $a_\Lambda : \Lambda \to \mathbb{Z}\times\mathbb{Z}$ be functions sending a point to the nearest lattice point. More concretely, suppose for $m, n \in \mathbb{Z}$ and $0 < a, b < 1$,

$$a_\Gamma(x, w) = a_\Gamma(m + a, n + b) = \begin{cases} (m, n) & \text{if } a < 1/2, b < 1/2; \\ (m, n + 1) & \text{if } a < 1/2, b \geq 1/2; \\ (m + 1, n) & \text{if } a \geq 1/2, b < 1/2; \\ (m + 1, n + 1) & \text{if } a \geq 1/2, b \geq 1/2. \end{cases}$$

Then $\mathcal{G}(\gamma, \Gamma) = \{T_x M_w \gamma : (x, w) \in \Gamma\}$ is symmetrically ℓ^1-localized with respect to $\mathcal{G}(\gamma, \Lambda) = \{T_y M_z \gamma : (z, y) \in \Lambda\}$, where

$$r = (r_{(m,n)})_{\mathbb{Z}\times\mathbb{Z}} = (2^{-1/2} e^{-\pi((m-1)^2+(n-1)^2)/2})_{\mathbb{Z}\times\mathbb{Z}}.$$

3 Equivalence Class Structure

The symmetries in this definition allow for a natural equivalence class structure, as was shown for Fornasier's intrinsically localized frames in [13].

Definition 3.1. *Let $S^1 := \{\mathcal{F} = (f_x)_{x\in X} \,|\, \mathcal{F}$ is a symmetrically ℓ^1-self-localized frame of $\mathcal{H}\}$. For $\mathcal{F}, \mathcal{E} \in S^1$, we define the relation $\mathcal{F} \sim \mathcal{E}$ if $\mathcal{F}$ is symmetrically ℓ^1-localized with respect to $\mathcal{E}$.*

We have the following theorem.

Theorem 3.2. [5] *Let $\mathcal{F} \in S^1$. Then for $\tilde{\mathcal{F}}$, $\tilde{\mathcal{F}} \in S^1$ and $\mathcal{F} \sim \tilde{\mathcal{F}}$.*

This theorem applies directly as our definition coincides with that of BCHL in the case of self-localization.

Before verifying that we have an equivalence relation, let us first prove the following proposition.

Proposition 3.3. *Let $\mathcal{F} = (f_x)_{x\in X}$ and $\mathcal{E} = (e_y)_{y\in Y}$ be frame sequences for Hilbert space $\mathcal{H}$, X, and Y arbitrary index sets. Let $a_X : X \to G$, $a_Y : Y \to G$ be associated maps. Suppose the following conditions are satisfied:*

1. *$\mathcal{F}$ is symmetrically ℓ^1-localized with respect to $\mathcal{E}$, i.e., there exists $r \in \ell^1(G)$ such that $|\langle f_x, e_y\rangle| \le r_{a_X(x)-a_Y(y)}$.*
2. *$\mathcal{F}$ is symmetrically ℓ^1-localized with respect to $\tilde{\mathcal{E}}$, i.e., there exists $s \in \ell^1(G)$ such that $|\langle f_x, \tilde{e}_y\rangle| \le s_{a_X(x)-a_Y(y)}$.*

Then $\mathcal{F} \in S^1$ and $\mathcal{E} \in S^1$.

Proof. We define convolution for $\ell^p(G)$ in the following way:

$$(c_j)_{j\in G} = (b_j)_{j\in G} * (d_j)_{j\in G} = \left(\sum_{k\in G} b_k d_{j-k}\right)_{j\in G}.$$

$$
\begin{aligned}
|\langle f_x, f_z\rangle| &= \left|\left\langle \sum_{y\in Y} \langle f_x, e_y\rangle \tilde{e}_y, f_z \right\rangle\right| \\
&\leq \sum_{y\in Y} |\langle f_x, e_y\rangle|\, |\langle \tilde{e}_y, f_z\rangle| \\
&\leq \sum_{y\in Y} r_{a_X(x)-a_Y(y)}\ s_{a_Y(y)-a_X(z)} \\
&= \sum_{j\in G} \sum_{y\in a_Y^{-1}(j)} r_{a_X(x)-a_Y(y)}\ s_{a_Y(y)-a_X(z)} \\
&\leq \sum_{j\in G} K r_{a_X(x)-j}\ s_{j-a_X(z)} \\
&= K(r*s)_{a_X(x)-a_X(z)}.
\end{aligned}
$$

As $r, s \in \ell^1(G)$, we have that $r * s \in \ell^1(G)$. Hence, $\mathcal{F} \in S^1$. By Theorem 3.2, $\tilde{\mathcal{F}} \in S^1$, so there is a $q \in \ell^1(G)$ such that $|\langle \tilde{f}_x, \tilde{f}_z\rangle| \leq q_{a_X(x)-a_X(z)}$. Then by a similar calculation as above,

$$|\langle e_y, \tilde{f}_x\rangle| \leq \sum_{z\in X} |\langle e_y, f_z\rangle|\, |\langle \tilde{f}_z, \tilde{f}_x\rangle| \leq K(r*q)_{a_Y(y)-a_X(x)}.$$

Finally,

$$|\langle e_y, e_z\rangle| \leq \sum_{x\in X} |\langle e_y, \tilde{f}_x\rangle|\, |\langle f_x, e_z\rangle| \leq K(r*q*r)_{a_Y(y)-a_Y(z)}.$$

As $r, q \in \ell^1(G)$, $r*q*r \in \ell^1(G)$. Hence, $\mathcal{E} \in S^1$. □

Proposition 3.4. *Let $\mathcal{F} = (f_x)_{x\in X}$ and $\mathcal{E} = (e_y)_{y\in Y}$ be frame sequences for Hilbert space $\mathcal{H}$, X, and Y arbitrary index sets. Let $a_X : X \to G$, $a_Y : Y \to G$ be associated maps. Suppose the following are satisfied,*

(1) $\mathcal{E} \in S^1$,

(2) *$\mathcal{F}$ is symmetrically ℓ^1-localized with respect to $\mathcal{E}$, i.e., there exists $r \in \ell^1(G)$ such that $|\langle f_x, e_y\rangle| \leq r_{a_X(x)-a_Y(y)}$.*

Then $\mathcal{F}$ is ℓ^1 localized with respect to $\tilde{\mathcal{E}}$, and $\mathcal{F} \in S^1$.

Proof. By theorem 3.2, if $\mathcal{E} \in S^1$, then $\tilde{\mathcal{E}} \in S^1$. So let $s \in \ell^1(G)$ such that $|\langle \tilde{e}_y, \tilde{e}_z\rangle| \leq s_{a_Y(y)-a_Y(z)}$. If $K = \sup_{j\in G} |a_Y^{-1}(j)|$, then

$$\begin{aligned}
|\langle f_x, \tilde{e}_y\rangle| &\le \sum_{z\in Y} |\langle f_x, e_z\rangle|\,|\langle \tilde{e}_z, \tilde{e}_y\rangle| \\
&\le \sum_{z\in Y} r_{a_X(x)-a_Y(z)}\, s_{a_Y(z)-a_Y(y)} \\
&= \sum_{j\in G} \sum_{z\in a_Y^{-1}(j)} r_{a_X(x)-a_Y(z)}\, s_{a_Y(z)-a_Y(y)} \\
&\le \sum_{j\in G} K r_{a_X(x)-j}\, s_{j-a_Y(y)} \\
&= K(r*s)_{a_X(x)-a_Y(y)}.
\end{aligned}$$

As $r,s\in\ell^1(G), r*s\in\ell^1(G)$. Hence, $\mathcal{F}$ is ℓ^1 localized with respect to $\tilde{\mathcal{E}}$. So by Proposition 3.3, $\mathcal{F}\in S^1$. □

Theorem 3.5. *$\sim$ is an equivalence relation on S^1.*

Proof. Reflexivity: By definition, $\mathcal{F}\sim\mathcal{F}$.

Symmetry: It is clear to see that $\mathcal{F}\sim\mathcal{E}\Rightarrow\mathcal{E}\sim\mathcal{F}$.

Transitivity: Let $\mathcal{F}=(f_x)_{x\in X}, \mathcal{E}=(e_y)_{y\in Y}, \mathcal{G}=(g_z)_{z\in Z}\in S^1$, such that $\mathcal{F}\sim\mathcal{E}$ and $\mathcal{E}\sim\mathcal{G}$. Let $K=\sup_{j\in G}|a_Y^{-1}(j)|$. By Proposition 3.4 and symmetry, we have that $\tilde{\mathcal{E}}\sim\mathcal{G}$. Let $r\in\ell^1(G)$ and $s\in\ell^1(G)$ be such that $|\langle f_x, e_y\rangle|\le r_{a_X(x)-a_Y(y)}$ and $|\langle \tilde{e}_y, g_z\rangle|\le s_{a_Y(y)-a_Z(z)}$. Then we have

$$\begin{aligned}
|\langle f_x, g_z\rangle| &\le \sum_{y\in Y} |\langle f_x, e_y\rangle|\,|\langle \tilde{e}_y, g_z\rangle| \\
&\le \sum_{y\in Y} r_{a_X(x)-a_Y(y)}\, s_{a_Y(y)-a_Z(z)} \\
&= \sum_{j\in G} \sum_{y\in a_Y^{-1}(j)} r_{a_X(x)-a_Y(y)}\, s_{a_Y(y)-a_Z(z)} \\
&= \sum_{j\in G} K r_{a_X(x)-j}\, s_{j-a_Z(z)} \\
&\le K(r*s)_{a_X(x)-a_Z(z)}.
\end{aligned}$$

Notice that $r,s\in\ell^1(G)$, so $r*s\in\ell^1(G)$. Hence, $\mathcal{F}\sim\mathcal{G}$. □

From this equivalence class structure we obtain the following.

Corollary 3.6. *For all $\mathcal{F},\mathcal{E}\in S^1$, $\mathcal{F}\sim\mathcal{E}$ implies $\mathcal{F}\sim\tilde{\mathcal{E}}, \tilde{\mathcal{F}}\sim\mathcal{E}, \tilde{\mathcal{F}}\sim\tilde{\mathcal{E}}$.*

Example 3.7. *Gabor frames for $L^2(\mathbb{R}^d)$ are modulations and translations of a single function, called an atom. These atoms ought to have good decay in both time and frequency, and a class of functions with such a property is the modulation space, M^1. M^1 consists of all functions f such that the short-time Fourier transform $V_g f : \mathbb{R}^{2d} \to \mathbb{C}$, defined $V_g f(x, \omega) = \langle f, M_\omega T_x g\rangle$, is in $L^1(\mathbb{R}^{2d})$. In [6] we have the following theorem.*

Theorem 3.8. [6] *Let $\mathcal{G}(\gamma, \Gamma) = (T_x M_w \gamma : (x, w) \in \Gamma)$ and $\mathcal{G}(\lambda, G) = (T_y M_z \lambda : (z, y) \in G)$ be Gabor frames in $L^2(\mathbb{R})$, where $\gamma, \lambda \in M^1$ and $G = \alpha\mathbb{Z}^d \times \beta\mathbb{Z}^d$. Then the following statements hold:*

(a) $\mathcal{G}(\gamma, \Gamma), \mathcal{G}(\lambda, G) \in S^1$,

(b) $\mathcal{G}(\gamma, \Gamma) \sim \mathcal{G}(\lambda, G)$.

Using this theorem and Corollary 3.6, we have that all Gabor frames with generators in the modulation space M^1 and their canonical duals, regardless of whether or not their indices have a lattice structure, are in the same equivalence class.

As previously mentioned, a similar relation was given in [13] and [14], which was brought to the attention of the author after this paper was nearly completed. Their relation was almost an equivalence relation, and was defined on the set of frames whose Gramian matrices lie in a solid, inverse closed, involutive Banach algebra. In contrast, the set of Gramian matrices of ℓ^1-self localized frames forms an algebra, but not necessarily an inverse closed Banach algebra. However, a frame $\mathcal{F}$ is ℓ^1-self localized if and only if a frame $\mathcal{F}'$ is localized in the sense of Fornasier and Gröchenig, i.e., the Gramian matrix is in the Sjöstrand algebra (see Theorem A.1, Remark A.2, and Lemma A.1 in [5]).

4 Extending the Results of Gröchenig

Gröchenig had two main results in [20], that a frame is localized with respect to a Riesz basis if and only if its canonical dual exhibits the same localization property, and frames localized with respect to a Riesz basis are automatically Banach frames for the family of Banach spaces naturally associated to the Riesz basis. We shall extend both of his results as a consequence of the equivalence relation.

4.1 Localization of the Dual

In Gröchenig's definition we have that a frame $\mathcal{F}$ is s-localized with respect to a Riesz basis $\mathcal{E}$ if we have polynomial decay with respect to both $\mathcal{E}$ and its canonical dual, $\tilde{\mathcal{E}}$. Since we have that s-localization implies symmetric ℓ^p-localization with $p > \frac{1}{s}$, assume $s > 1$. Then we have that $\mathcal{F}$ is symmetrically

ℓ^1-localized with respect to $\mathcal{E}$. By Proposition 3.3, if $\mathcal{F} = (f_x)_{x\in X}$ is s-localized with respect to a Riesz basis $\mathcal{E} = (e_y)_{y\in Y}$, $s > 1$, we have $\mathcal{F} \in S^1$ and $\mathcal{E} \in S^1$. So by Corollary 3.6, if $\mathcal{F} = (f_x)_{x\in X}$ is s-localized with respect to a Riesz basis $\mathcal{E} = (e_y)_{y\in Y}$, $s > 1$, then as $\mathcal{F} \sim \mathcal{E}$, we have $\tilde{\mathcal{F}} \sim \mathcal{E}$.

4.2 Construction of Banach Frames

One of the main goals regarding Banach frames is their construction. In particular, we would like to show that a self-localized frame defines a family of Banach spaces, and any other frame localized with respect to it is also a Banach frame for these spaces. As an example, consider their significance in the study of Gabor frames. Here, the Banach spaces H^p_m associated with a Gabor frame are the all-important modulation spaces [14]. We shall prove that if $\mathcal{F} = (f_x)_{x\in X}, \mathcal{E} = (e_y)_{y\in Y} \in S^1$ and $\mathcal{F} \sim \mathcal{E}$, then $\mathcal{F}$ is a Banach frame for the natural family of Banach spaces associated with $\mathcal{E}$. Note that when dealing with weights we need extra conditions on the indices for everything to make sense.

Definition 4.1. *Let $(B, \|\cdot\|_B)$ be a Banach space and let $(B_d(X), \|\cdot\|_{B_d})$be a Banach space of sequences indexed by X. A (countable) subset $(f_x : x \in X)$ of B', the dual of B, is called a* ***Banach frame*** *for B if the following properties hold:*

(a) The coefficient operator $C_{\mathcal{E}} : B \to B_d(X)$ defined by $C_{\mathcal{E}} f = (f_x(f))_{x\in X}$ is bounded.

(b) We have the norm equivalence $\|f\|_B \asymp \|f_x(f)\|_{B_d}$.

(c) There exist a bounded operator $R : B_d(X) \to B$, called the reconstruction operator, such that $R((f_x(f))_{x\in X}) = f$.

Definition 4.2. *For $1 \le p < \infty$, the* ***weighted ℓ^p-space*** *$\ell^p_m(Y)$ on the index set $Y \subset \mathbb{R}^d$ is defined by the norm*

$$\|c\|_{\ell^p_m} = \left(\sum_{y\in Y} |c_y|^p m(y)^p \right)^{1/p}$$

with the usual modification for $p = \infty$. The weight m is a non-negative function on $\mathbb{R}^d$, which we may assume without loss of generality is continuous.

For the purposes of this paper, we assume that the weight is submultiplicative, i.e., $m(j + k) \le m(j)m(k)$ for all $j, k \in \mathbb{R}^d$.

Definition 4.3. *Let $\mathcal{E} = (e_y)_{y\in Y}$ be a frame for $\mathcal{H}$ such that $\mathcal{E} \in S^1$ and $\tilde{\mathcal{E}}$ be the canonical dual frame. Let $\mathcal{H}_0 \subset \mathcal{H}$ be the subspace of finite linear combinations* *of elements in $\mathcal{E}$. For $0 < p < \infty$ and m a weight function, we define a (quasi-) norm on $\mathcal{H}_0$ by*

$$\|f\|_{\mathcal{H}^p_m} = \|(f, \tilde{e}_y)_{y\in Y}\|_{\ell^p_m}.$$

For $1 \le p < \infty$, the ***associated space*** *$\mathcal{H}^p_m(\mathcal{E}, \tilde{\mathcal{E}})$ is defined to be the norm completion of $\mathcal{H}_0$ with the norm $\|\cdot\|_{\mathcal{H}^p_m}$. For $p = \infty$, $\mathcal{H}^\infty_m(\mathcal{E}, \tilde{\mathcal{E}})$ is defined to be the completion of $\mathcal{H}_0$ in the $\sigma(\mathcal{H}, \mathcal{H}_0)$-topology.*

$\mathcal{H}^p_m(\mathcal{E}, \tilde{\mathcal{E}})$ is a Banach space for $1 \le p \le \infty$ with $\|f\|_{\mathcal{H}^p_m} \asymp \inf\{\|c\|_{\ell^p_m} : c \in \ell^p_m, f = \sum_{y\in Y} c_y e_y\}$, as proved in [14] using the following lemma.

Lemma 4.4. *Let $\mathcal{F}$ be symmetrically ℓ^1-localized with respect to $\mathcal{E}$ and consider the cross-Gramian matrix $A = (\langle e_y, f_x\rangle)_{y\in Y, x\in X}$. Let c be a finite sequence, then A acts on c in the following way: $(Ac)_{x\in X} = (\sum_{y\in Y}\langle e_y, f_x\rangle c_y)_{x\in X}$. Then A extends to a bounded operator from $\ell^p_m(Y)$ to $\ell^p_m(X)$, where m is a submultiplicative weight. If $m = 1$, then X, Y can be arbitrary countable indices. If $m \neq 1$, then we assume $X, Y, G \subset \mathbb{R}^d$, $m : \mathbb{R}^d \to \mathbb{R}$ and the maps $a_X : X \to G, a_Y : Y \to G$ coming from the localization are such that $\max\{\sup_{x\in X}|x - a_X(x)|, |y - a_Y(y)|\} \le \mu$ for some $\mu > 0$.*

Proof. Let $c \in \ell^p_m(Y)$. Define $d = (d_j)_{j\in G}$, where $d_j = \sum_{y\in a_Y^{-1}(j)}|c_y|$. Then $d \in \ell^p_m(G)$ as $|a_Y^{-1}(j)| \le K$ for all $j \in G$. Hence,

$$\begin{aligned}
|(Ac)_x| &= |\sum_{y\in Y}\langle e_y, f_x\rangle c_y| \\
&\le \sum_{y\in Y}|\langle e_y, f_x\rangle|\,|c_y| \\
&\le \sum_{j\in G}\sum_{y\in a_Y^{-1}(j)} r_{a_X(x)-a_Y(y)}\,|c_y| \\
&= \sum_{j\in G} r_{a_X(x)-j}\,d_j \\
&= (r * d)_{a_X(x)}.
\end{aligned}$$

Hence, by a proof found in [1], there exists some constant C such that

$$\|Ac\|_{\ell^p_m(X)} \le \|r * d\|_{\ell^p_m(G)} \le C\,\|r\|_{\ell^1_m(G)}\,\|d\|_{\ell^p_m(G)} < \infty.$$

We now prove that $\|d\|_{\ell^p_m(G)} \le MK\|c\|_{\ell^p_m(Y)}$, where $M = \sup_{|z|\le\mu} m(z)$ and $K = \max\{\sup_{j\in G} a_X^{-1}(j), \sup_{j\in G} a_Y^{-1}(j)\}$. Notice that as m is assumed to be continuous we have that M is finite.

For $1 \leq p < \infty$, since $\left(\sum_{y \in a_Y^{-1}(j)} |c_y|\right)^p \leq K^p \sum_{y \in a_Y^{-1}(j)} |c_y|^p$ and m is submultiplicative, we have

$$\begin{aligned}
\|d\|^p_{\ell^p_m(G)} &= \sum_{j \in G} d_j^p m(j)^p \\
&\leq K^p \sum_{j \in G} \sum_{y \in a_Y^{-1}(j)} |c_y|^p \, m(j)^p \\
&\leq K^p \sum_{j \in G} \sum_{y \in a_Y^{-1}(j)} |c_y|^p \, m(y)^p \, m(j-y)^p \\
&\leq M^p \, K^p \sum_{j \in G} \sum_{y \in a_Y^{-1}(j)} |c_y|^p \, m(y)^p \\
&= M^p \, K^p \, \|c\|^p_{\ell^p_m(Y)}.
\end{aligned}$$

For $p = \infty$,

$$\begin{aligned}
\|d\|_{l^\infty_m(G)} &= \operatorname*{ess\,sup}_{j \in G} \; d_j \, m(j) \\
&= \operatorname*{ess\,sup}_{j \in G} \sum_{y \in a_Y^{-1}(j)} |c_y| \, m(j) \\
&\leq \operatorname*{ess\,sup}_{j \in G} \sum_{y \in a_Y^{-1}(j)} |c_y| \, m(y) \, m(j-y) \\
&\leq \sum_{y \in a_Y^{-1}(j)} \operatorname*{ess\,sup}_{y \in Y} |c_y| \, m(y) \, M \\
&\leq M \, K \, \|c\|_{l^\infty_m(Y)}.
\end{aligned}$$

Hence, $\|Ac\|_{\ell^p_m(X)} \leq M \, K \, C \, \|r\|_{\ell^1_m(G)} \, \|c\|_{\ell^p_m(Y)}$. □

As noted in [14], The elements of $\mathcal{H}^p_m$ are technically equivalence classes of Cauchy sequences of elements of $\mathcal{H}_0$. Though $\mathcal{H}^p_m$ may happen to be $\{0\}$ or dependent on the choice of duals, if the frame is ℓ^1-self-localized, then $\mathcal{H}^p_m$ is a Banach space independent of the choice of a dual and all self-localized frames in a particular equivalence class will be Banach frames for the same space.

Theorem 4.5. *Suppose $\mathcal{F} = (f_x)_{x \in X}$ and $\mathcal{E} = (e_y)_{y \in Y}$ are frames for a Hilbert space $\mathcal{H}$, $\mathcal{F}, \mathcal{E} \in S^1$. If $\mathcal{F} \sim \mathcal{E}$, then $\mathcal{F}$ is a Banach frame for the family of Banach spaces $\mathcal{H}^p(\mathcal{E}, \tilde{\mathcal{E}})$.*

Proof. We will need to satisfy the following conditions:

(a) The coefficient operator $C : \mathcal{H}^p \to \ell^p(X)$ defined $Cf = (\langle f, f_x \rangle)_{x \in X}$ is bounded.

(b) There exists a bounded operator R from $\ell^p(X)$ to $\mathcal{H}^p$, called the reconstruction operator, such that $R((\langle f, f_x\rangle)_{x\in X}) = f$.

(c) We have the norm equivalence $\|f\|_{\mathcal{H}^p} \asymp \|\langle f, f_x\rangle\|_{\ell^p(X)}$.

Let $B = \mathcal{H}^p$ and $B_d = \ell^p(X)$, where $1 \le p \le \infty$. If $f \in \mathcal{H}^p \subseteq \mathcal{H}$, then $f = \sum_{y\in Y} c_y e_y$ where $c \in \ell^p(Y)$. We define the linear functionals $(f_x)_{x\in X} \in (\mathcal{H}^p)'$ in the following way:

$$f_x(f) = \langle f, f_x\rangle \ = \sum_{y\in Y} c_y \langle e_y, f_x\rangle.$$

Let $B = \sup_{y\in Y} |\langle e_y, f_x\rangle|$. Notice that f_x is bounded for each x:

$$|f_x(f)| = |\sum_{y\in Y} c_y\langle e_y, f_x\rangle| = \|Ac\|_{\ell^p(X)} \le \alpha||c||_{\ell^p(Y)} \le \beta||f||_{\mathcal{H}^p}.$$

If $\mathcal{H}^p \nsubseteq \mathcal{H}$, define f_x as above for $f = \sum_{y\in Y} c_y e_y$, *supp* $c < \infty$. f_x is still a bounded linear functional. Then by a corollary of the Hahn Banach theorem, we can extend f_x to a bounded linear functional on the Banach space $\mathcal{H}^p$ complete in the norm $||\cdot||_{\mathcal{H}^p}$.

Now let $f \in \mathcal{H}^p$; then there is a Cauchy sequence $(f_n)_{n\in\mathbb{N}}$ that converges to f, where $f_n = \sum_{y\in Y} c'_y e_y$, *supp* $c' < \infty$. This limit is unique. The boundedness of the linear functional gives us that $(f_x(f_n))_{n\in\mathbb{N}}$ is a Cauchy sequence which converges to $f_x(f)$ and this limit is also unique.

(a) Let C be the coefficient operator, $Cf := (\langle f, f_x\rangle)_{x\in X}$. First consider $f = \sum_{y\in Y} c_y e_y \in \mathcal{H}^p \subseteq \mathcal{H}$, where $c \in \ell^p(Y)$ and $||f||_{\mathcal{H}^p} \asymp ||c||_{\ell^p(Y)}$. Then $Cf = (\sum_{y\in Y} c_y\langle e_y, f_x\rangle)_{x\in X}$. We have

$$\|Cf\|^p_{\ell^p(X)} = \|Ac\|^p_{\ell^p(X)} \le \alpha\|f\|^p_{\mathcal{H}^p}$$

where A is defined as in Lemma 4.4, with $(Ac)_{x\in X} = \sum_{y\in Y} \langle e_y, f_x\rangle c_y$, and $m = 1$. Hence, the coefficient operator C is bounded from $\mathcal{H}^p$ to $\ell^p(X)$.

If $\mathcal{H}^p \nsubseteq \mathcal{H}$, then we can define C as above on finite sums. As C is continuous, C can be extended to a bounded linear operator on the completion.

(b) Let $c = (c_x)_{x\in X}$ be a finite sequence. Let the reconstruction operator R be the synthesis operator:

$$Rc = D_{\tilde{\mathcal{F}}} c = \sum_{x\in X} c_x \tilde{f}_x.$$

By a proof nearly identical to that of Proposition 2.4 in [14], we have that R is bounded on $\ell^p(X)$ for $1 \le p \le \infty$. If $f \in \mathcal{H}^p \subseteq \mathcal{H}$,

$$R((\langle f, f_x\rangle)_{x\in X}) = \sum_{x\in X} \langle \sum_{y\in Y} c'_y e_y, f_x\rangle \tilde{f}_x = \sum_{x\in X}\sum_{y\in Y} c'_y\langle e_y, f_x\rangle\tilde{f}_x = \sum_{y\in Y} c'_y e_y = f.$$

If $f \in \mathcal{H}^p \nsubseteq \mathcal{H}$, let $(f_n)_{n\in\mathbb{N}}$ be a Cauchy sequence in $\mathcal{H}^p$ converging to f, where $f_n = \sum_{y\in Y} c'_y e_y, supp\ c' < \infty$. Then $R((f_x(f_n))_{x\in X}) = f_n$. Recall that f_x is a bounded linear functional, so $f_x(f_n)$ converges to $f_x(f)$. As the limits are unique, we have that $R((f_x(f))_{x\in X}) = f$.

(c) We have that

$$||Cf||_{\ell^p(X)} = ||(\langle f, f_x\rangle)_{x\in X}||_{\ell^p(X)} \leq \alpha||f||_{\mathcal{H}^p}$$

and

$$||RCf||_{\mathcal{H}^p} = ||f||_{\mathcal{H}^p} \leq D||Cf||_{\ell^p(X)}.$$

Hence,

$$\frac{1}{\alpha}||(\langle f, f_x\rangle)_{x\in X}||_{\ell^p(X)} \leq ||f||_{\mathcal{H}^p} \leq D||(\langle f, f_x\rangle)_{x\in X}||_{\ell^p(X)}.$$

Hence, $\mathcal{F}$ is automatically a Banach frame for $\mathcal{H}^p$. □

If we add a weight, m, and consider the weighted ℓ^p space, ℓ^p_m, we run into the problem of having to define m for X, Y, and G. We deal with this problem by embedding X, Y, and G into a larger space S (this generalizes the case where X, Y, and G are subsets of $\mathbb{R}^d$) and adding the extra condition that the maps $a_X : X \to G, a_Y : Y \to G$ are such that $|x - a_X(x)|, |y - a_Y(y)| \leq \mu$ for some $\mu > 0$.

Theorem 4.6. *Suppose $\mathcal{F} = (f_x)_{x\in X}$ is a frame and let $\mathcal{E} = (e_y)_{y\in Y}$ be a Riesz basis for $\mathcal{H}$, $\mathcal{F}, \mathcal{E} \in S^1$, $X, Y \subset S$, where S is an abelian group. Assume $G \subset S$, and the maps $a_X : X \to G, a_Y : Y \to G$ are such that $|x - a_X(x)|, |y - a_Y(y)| \leq \mu$ for some $\mu > 0$. Let $m : S \to R^*_+$ be a submultiplicative weight function, i.e., a non-negative, locally integrable function on S such that for all $x, y \in S$, $m(x + y) \leq m(x)m(y)$. Assume without loss of generality that m is continuous and symmetric. If $\mathcal{F} \sim \mathcal{E}$, then $\mathcal{F}$ is a Banach frame for the family of Banach spaces $\mathcal{H}^p_m$ associated with $\mathcal{E}$.*

Proof. The proof is almost identical to that of the previous theorem. Note that, Lemma 4.5 still holds true if we replace $\mathbb{R}^d$ by S. □

ACKNOWLEDGEMENT

The author would like to gratefully acknowledge the conversations with Akram Aldroubi, Radu Balan, Peter G. Casazza, Massimo Fornasier, Karlheinz Gröchenig, and Christopher Heil. The author would also like to thank the referees for their helpful comments and suggestions.

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