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Sinc Approximation with a Gaussian Multiplier

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Abstract

Recently, it was shown with the help of Fourier analysis that by incorporating a Gaussian multiplier into the truncated classical sampling series, one can approximate bandlimited signals of finite energy with an error that decays exponentially as a function of the number of involved samples. Based on complex analysis, we show for a slightly modified operator that this approximation method applies not only to bandlimited signals of finite energy, but also to bandlimited signals of infinite energy, to classes of non-bandlimited signals, to *all* entire functions of exponential type (including those whose samples increase exponentially), and to functions analytic in a strip and not necessarily bounded. Moreover, the method extends to non-real argument. In each of these cases, the use of $2N+1$ samples results in an error bound of the form $M\mathrm{e}^{-\alpha N}$, where M and α are positive numbers that do not depend on N. The power of the method is illustrated by several examples.

Key words and phrases : sinc approximation, sampling series, Gaussian convergence factor, error bounds, entire functions of exponential type, functions analytic in a strip

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1 Introduction

Throughout this paper, we shall use the following notation. For $\sigma \geq 0$ we denote by $\mathcal{E}_\sigma$ the set of all function f which are *entire*, that is, analytic in the whole complex plane, and satisfy

$$\limsup_{r\to\infty} \frac{\log \max_{|z|=r} |f(z)|}{r} \le \sigma\,. \tag{1}$$

We call $\mathcal{E}_\sigma$ the class of entire functions of exponential type σ.

When we have a function $f : \mathbb{C} \to \mathbb{C}$ and write $f \in L^p(\mathbb{R})$, we actually mean that the restriction of f to $\mathbb{R}$ belongs to $L^p(\mathbb{R})$. The norm in $L^p(\mathbb{R})$ will be denoted by $\|\cdot\|_p$ for $p \in [1,\infty]$. The spaces $\mathcal{B}_\sigma^p := \mathcal{E}_\sigma \cap L^p(\mathbb{R})$ are called Bernstein spaces; see [5, § 6.1]. We have

$$\mathcal{B}_\sigma^1 \subset \mathcal{B}_\sigma^p \subset \mathcal{B}_\sigma^r \subset \mathcal{B}_\sigma^\infty \qquad (1 \le p \le r \le \infty).$$

It is well known that a function f is a signal of finite energy bandlimited to $[-\sigma,\sigma]$ if and only if f is the restriction to $\mathbb{R}$ of a function from $\mathcal{B}_\sigma^2$.

Introducing the sinc function by

$$\operatorname{sinc} z := \begin{cases} \dfrac{\sin \pi z}{\pi z} & \text{if} \quad z \in \mathbb{C}\setminus\{0\}, \\ 1 & \text{if} \quad z = 0, \end{cases}$$

we may state the classical sampling theorem of Whittaker–Kotel'nikov–Shannon (see [7, p. 49]) as follows.

Theorem A *Let $f \in \mathcal{B}_\sigma^2$, where $\sigma > 0$. Then, for $h \in (0,\pi/\sigma]$, we have*

$$f(z) = \sum_{n=-\infty}^{\infty} f(hn)\operatorname{sinc}(h^{-1}z - n) \qquad (z \in \mathbb{C}). \tag{2}$$

The series converges absolutely and uniformly on every strip $|\Im z| \le K$ for any $K > 0$.

In practice, the use of (2) is limited since the series converges slowly unless $|f(x)|$ decays rapidly as $x \to \pm\infty$. However, if we choose h strictly less than π/σ, which is the case of *oversampling*, we can easily incorporate a convergence factor into the series in (2). In fact, for $0 < \varepsilon < \pi - h\sigma$, let $\Phi \in \mathcal{E}_\varepsilon$ such that $\Phi(0) = 1$. Suppose that $|\Phi(z)|$ decays rapidly on lines parallel to the real axis, at least as fast as $O(|\Re z|^{-1})$, say. Then, for any fixed $\zeta \in \mathbb{C}$, the function

$$z \longmapsto f(z)\Phi\left(h^{-1}(\zeta - z)\right) \tag{3}$$

belongs to $\mathcal{B}^2_{\sigma+\varepsilon/h}$. Applying Theorem A, we obtain

$$f(z)\Phi\left(h^{-1}(\zeta - z)\right) = \sum_{n=-\infty}^{\infty} f(hn)\Phi(h^{-1}\zeta - n)\operatorname{sinc}(h^{-1}z - n). \tag{4}$$

Finally, substituting $\zeta = z$, we arrive at

$$f(z) \;=\; \sum_{n=-\infty}^{\infty} f(hn)\Phi(h^{-1}z - n)\,\mathrm{sinc}\,(h^{-1}z - n). \tag{5}$$

The speed of convergence of the series in (5) is determined by the decay of $|\Phi|$. However, the decay of an entire function of exponential type is limited. It is known (see [1], [6, p. 81]) that if

$$\Phi(x) \;=\; O\left(\mathrm{e}^{-w(|x|)}\right) \qquad (x \to \pm\infty),$$

where w is a non-negative function satisfying some mild regularity conditions on the positive real line, then we must have

$$\int_1^\infty \frac{w(x)}{x^2}\,\mathrm{d}x \;<\; \infty.$$

This shows that an entire function of exponential type cannot decay as fast as $\mathrm{e}^{-c|x|}$ for some positive c. As a consequence, by taking N terms of the series in (5), we cannot have a truncation error that is bounded by

$$M\mathrm{e}^{-\alpha N} \qquad (N \in \mathbb{N}) \tag{6}$$

for some positive constants M and α.

Now, let

$$G(z) \;:=\; \exp\left(-\,z^2\right), \tag{7}$$

which may be called a *Gaussian function* since, apart from a normalization, it describes the Gaussian probability distribution. It is not difficult to show that if in (5) we used an appropriately scaled form of G instead of Φ, we could obtain a truncation error that has a bound of the form (6). Unfortunately, G is not of exponential type and so (5) will no longer be valid.

Nevertheless, as was mentioned to one of us by Liwen Qian, Chinese chemists have used a Gaussian multiplier in the sampling series, without caring for a mathematical justification, and they obtained numerical results of high accuracy.

A rigorous mathematical investigation of the effect of a Gaussian multiplier was given by Qian, Qian and Creamer, and Qian and Ogawa in a series of papers where the authors considered the classical sampling theorem and various variants (see [8], [9], [10], [11], [12]). We can say that if Φ is replaced by the Gaussian function, then (5) holds with an error term which occurs as an aliasing error in the above derivation while the cited authors called it a *localization error*. It turns out that, by scaling G appropriately, this additional error can be made at least as small as (6). Hence, the total error has also a bound of the form (6).

For a precise statement, we denote by $\lfloor x \rfloor$, where $x \in \mathbb{R}$, the largest integer not exceeding x. Then a result in [9, Theorem 2.2 for $b = 1$, $s = 0$] may be formulated as follows.

Theorem B *Let* $f \in \mathcal{B}_\sigma^2$, *where* $\sigma > 0$. *For* $h \in (0, \pi/\sigma)$, $\alpha := (\pi - h\sigma)/2$, $N \in \mathbb{N} \setminus \{1, 2\}$, $x \in \mathbb{R}$, *and*

$$Z_N(x) := \left\{n \in \mathbb{Z} : |\lfloor x \rfloor - n| \leq N\right\},$$

define

$$\mathcal{G}_{h,N}[f](x) := \sum_{n \in Z_N(x/h)} f(hn) \operatorname{sinc}\left(\frac{x}{h} - n\right) \exp\left(-\frac{\alpha}{N-2}\left(\frac{x}{h} - n\right)^2\right). \qquad (8)$$

Then

$$\left|f(x) - \mathcal{G}_{h,N}[f](x)\right| \leq \frac{2\mathrm{e}^{-\alpha(N-2)}}{\sqrt{\pi\alpha(N-2)}} \sqrt{\frac{\sigma}{\pi}} \, \|f\|_2 \left[1 + \frac{\mathrm{e}^{3\alpha/(N-2)}}{2\sqrt{\pi\alpha(N-2)}}\right]. \qquad (9)$$

The proofs in the cited papers by Qian and his co-authors are always based on real Fourier analysis. Therefore, the condition that f belongs to $L^2(\mathbb{R})$ cannot be considerably relaxed. Moreover, the proofs do not extend to non-real argument of f.

In this paper, we show that sinc approximation with a Gaussian convergence factor can be conveniently studied by using the method of contour integration. This approach has the following advantages:

- it is simpler;
- it gives slightly better error bounds;
- it extends to non-real argument;
- it extends to functions that do not belong to $L^2(\mathbb{R})$; even unbounded functions can be considered;
- it admits functions that are not restrictions of entire functions;
- it extends to certain classes of non-bandlimited signals;
- it is very flexible and allows various modifications.

Our investigation was started in July 2000 when the second named author spent a few days in Erlangen but it was delayed afterwards and gained a new impulse through the work of Qian Liwen et al.

2 Entire Functions

In the following theorem the hypothesis on f is much weaker than in Theorem B and the conclusion extends to non-real argument. Note that the operator (11) used for approximating f has been slightly modified as compared with (8). The truncation has been made symmetric. This avoids that after fixing N, the error bound is obtained in terms of $N-2$.

Theorem 2.1 *Let f be an entire function such that*

$$|f(\xi+\mathrm{i}\eta)| \le \phi(|\xi|)\,\mathrm{e}^{\sigma|\eta|} \qquad (\xi,\,\eta\in\mathbb{R}), \tag{10}$$

where ϕ is a non-decreasing, non-negative function on $[0,\infty)$ and $\sigma\ge 0$. For $h\in(0,\pi/\sigma)$, $\alpha := (\pi-h\sigma)/2$, $N\in\mathbb{N}$, $z\in\mathbb{C}$, $|\Im z|<N$ and $N_z := \lfloor \Re z+\frac12\rfloor$, define

$$\mathcal{C}_{h,N}[f](z) := \sum_{n=N_{z/h}-N}^{N_{z/h}+N} f(hn)\,\mathrm{sinc}\left(\frac{z}{h}-n\right)\exp\left(-\frac{\alpha}{N}\left(\frac{z}{h}-n\right)^2\right). \tag{11}$$

Then

$$\bigl|f(z)-\mathcal{C}_{h,N}[f](z)\bigr| \le \bigl|\sin(h^{-1}\pi z)\bigr|\,\frac{2\mathrm{e}^{-\alpha N}}{\sqrt{\pi\alpha N}}\,\phi\bigl(|\Re z|+h(N+1)\bigr)\,\beta_N(h^{-1}\Im z), \tag{12}$$

where

$$\begin{aligned}\beta_N(t) \;&:=\; \cosh(2\alpha t)+\frac{2\mathrm{e}^{\alpha t^2/N}}{\sqrt{\pi\alpha N}[1-(t/N)^2]}+\frac12\left[\frac{\mathrm{e}^{2\alpha t}}{\mathrm{e}^{2\pi(N-t)}-1}+\frac{\mathrm{e}^{-2\alpha t}}{\mathrm{e}^{2\pi(N+t)}-1}\right]\\ &=\; \cosh(2\alpha t)+O(N^{-1/2}) \qquad (N\to\infty).\end{aligned}$$

Proof Let $z=x+\mathrm{i}y$, where $x,y\in\mathbb{R}$. We may assume that $\sigma<\pi$ and $h=1$. The more general result can be deduced from that special case by considering the function $\zeta\mapsto f(h\zeta)$. We may also assume that $y\ge 0$. The more general result can be deduced from that special case by considering the function $\zeta\mapsto\overline{f(\overline{\zeta})}$.

After these specializations, we introduce $N' := N+\frac12$ and denote by $\mathcal{R}$ the positively oriented rectangle with vertices at $\pm N'+N_z+\mathrm{i}(y\pm N)$. Now, setting $\omega := \alpha/N$, we consider the kernel function

$$K(z,\zeta) := \frac{\sin\pi z}{2\pi\mathrm{i}}\cdot\frac{f(\zeta)\,G(\sqrt{\omega}(z-\zeta))}{(\zeta-z)\sin\pi\zeta}\,. \tag{13}$$

When z is not an integer, then, as a function of ζ, this kernel has simple poles at z and at the integers, and has no other singularities. By the residue theorem, we readily find that

$$E := f(z) - \mathcal{C}_{1,N}[f](z) = \int_{\mathcal{R}} K(z,\zeta)\,\mathrm{d}\zeta$$
$$= \frac{\sin \pi z}{2\pi \mathrm{i}} \int_{\mathcal{R}} \frac{f(\zeta)\, G(\sqrt{\omega}(z-\zeta))}{(\zeta - z)\sin \pi\zeta}\,\mathrm{d}\zeta\,.$$

Now, denote by $I_{\mathrm{hor}}^{\pm}$ the contributions to the last integral coming from the two horizontal parts of $\mathcal{R}$, where $+$ and $-$ refer to the upper and the lower line segment, respectively. Similarly, denote by $I_{\mathrm{vert}}^{\pm}$ the contributions coming from the two vertical parts of $\mathcal{R}$, where $+$ and $-$ refer to the right and the left line segment, respectively. Then

$$E = \frac{\sin \pi z}{2\pi \mathrm{i}} \left(I_{\mathrm{hor}}^{-} + I_{\mathrm{vert}}^{+} + I_{\mathrm{hor}}^{+} + I_{\mathrm{vert}}^{-} \right), \tag{14}$$

where

$$I_{\mathrm{hor}}^{\pm} = \mp \int_{-N'+N_z}^{N'+N_z} \frac{f(t+\mathrm{i}(y \pm N))\, G(\sqrt{\omega}(x - t \mp \mathrm{i}N))}{(t - x \pm \mathrm{i}N)\sin \pi(t + \mathrm{i}(y \pm N))}\,\mathrm{d}t$$

and

$$I_{\mathrm{vert}}^{\pm} = \pm\mathrm{i} \int_{-N+y}^{N+y} \frac{f(\pm N' + N_z + \mathrm{i}t)\, G(\sqrt{\omega}(z \mp N' - N_z - \mathrm{i}t))}{(\pm N' + N_z + \mathrm{i}t - z)\sin \pi(\pm N' + N_z + \mathrm{i}t)}\,\mathrm{d}t\,.$$

In order to estimate these integrals, we use the following inequalities holding for $\xi, \eta, t \in \mathbb{R}$:

$$|G(\xi + \mathrm{i}\eta)| \le \mathrm{e}^{-\xi^2} \cdot \mathrm{e}^{\eta^2}, \tag{15}$$

$$|\sin(\xi + \mathrm{i}\eta)| \ge |\sinh \eta| = \frac{\mathrm{e}^{|\eta|}}{2}\left(1 - \mathrm{e}^{-2|\eta|}\right), \tag{16}$$

$$\left|\sin \pi(\pm N' + N_z + \mathrm{i}t)\right| = \cosh \pi t \ge \frac{\mathrm{e}^{\pi|t|}}{2}, \tag{17}$$

and

$$\left|\pm N' + N_z + \mathrm{i}t - z\right| \ge N\,. \tag{18}$$

Furthermore, for any point $\xi + \mathrm{i}\eta \in \mathcal{R}$, we have

$$|f(\xi + \mathrm{i}\eta)| \le \phi(N' + |N_z|)\mathrm{e}^{\sigma|\eta|} \le \phi(|x| + N + 1)\mathrm{e}^{\sigma|\eta|}.$$

With these estimates, we find that

$$\left|I_{\mathrm{hor}}^{\pm}\right| \le \frac{2\phi(|x|+N+1)\mathrm{e}^{\sigma|y\pm N|}\mathrm{e}^{-\pi|y\pm N|}\mathrm{e}^{\omega N^2}}{N\,(1 - \mathrm{e}^{-2\pi|y\pm N|})} \int_{-N'+N_z}^{N'+N_z} \mathrm{e}^{-\omega(x-t)^2}\,\mathrm{d}t$$
$$\le \frac{2\phi(|x|+N+1)\mathrm{e}^{-\alpha N}\mathrm{e}^{\mp 2\alpha y}}{N\,(1 - \mathrm{e}^{-2\pi|y\pm N|})} \int_{-\infty}^{\infty} \mathrm{e}^{-\omega s^2}\,\mathrm{d}s$$
$$= 2\pi\phi(|x|+N+1)\frac{\mathrm{e}^{-\alpha N}}{\sqrt{\pi\alpha N}} \cdot \frac{\mathrm{e}^{\mp 2\alpha y}}{1 - \mathrm{e}^{-2\pi|y\pm N|}},$$

and so

$$\begin{aligned} \left|I_{\mathrm{hor}}^{+}\right| + \left|I_{\mathrm{hor}}^{-}\right| &\leq 2\pi\phi(|x|+N+1)\frac{\mathrm{e}^{-\alpha N}}{\sqrt{\pi\alpha N}} \\ &\quad \times \left[2\cosh(2\alpha y) + \frac{\mathrm{e}^{2\alpha y}}{\mathrm{e}^{2\pi(N-y)}-1} + \frac{\mathrm{e}^{-2\alpha y}}{\mathrm{e}^{2\pi(N+y)}-1}\right]. \end{aligned} \tag{19}$$

For the contributions coming from the vertical parts of $\mathcal{R}$, we have

$$\begin{aligned} \left|I_{\mathrm{vert}}^{\pm}\right| &\leq 2\phi(|x|+N+1)\frac{\mathrm{e}^{-\omega N^2}}{N}\int_{-N+y}^{N+y} \mathrm{e}^{-(\pi-\sigma)|t|+\omega(y-t)^2}\,\mathrm{d}t \\ &= 2\phi(|x|+N+1)\frac{\mathrm{e}^{-\alpha N}}{N}\int_{-N}^{N} \mathrm{e}^{-2\alpha|s+y|+\omega s^2}\,\mathrm{d}s\,. \end{aligned}$$

In order to estimate the last integral, we determine a piecewise linear majorant for the exponent of e. Using a convexity property of parabolas, we find that

$$-2\alpha\,|s+y| + \omega s^2 \leq \begin{cases} \alpha[y+(1-y/N)s] & \text{if } s \in [-N,-y], \\ -\alpha[y+(1+y/N)s] & \text{if } s \in [-y,N]. \end{cases}$$

With these majorants we obtain

$$\int_{-N}^{-y} \mathrm{e}^{-2\alpha|s+y|+\omega s^2}\,\mathrm{d}s < \frac{\mathrm{e}^{\alpha y^2/N}}{\alpha(1-y/N)}$$

and

$$\int_{-y}^{N} \mathrm{e}^{-2\alpha|s+y|+\omega s^2}\,\mathrm{d}s < \frac{\mathrm{e}^{\alpha y^2/N}}{\alpha(1+y/N)}\,.$$

Hence,

$$\left|I_{\mathrm{vert}}^{\pm}\right| < 4\phi(|x|+N+1)\frac{\mathrm{e}^{-\alpha N}}{\alpha N}\cdot\frac{\mathrm{e}^{\alpha y^2/N}}{1-(y/N)^2}. \tag{20}$$

Finally, combining (14), (19), and (20), we arrive at the desired error bound in the case where $h = 1$. □

Note that in (11) the summation depends on the real part of z. Globally, $\mathcal{C}_{h,N}[f]$ provides a piecewise analytic approximation. On each of the strips

$$\left\{z \in \mathbb{C} \,:\, \left(k - \frac{1}{2}\right)h \leq \Re z < \left(k + \frac{1}{2}\right)h\right\} \qquad (k \in \mathbb{Z}),$$

$\mathcal{C}_{h,N}[f]$ is the restriction of an entire function of order two.

In Theorem 2.1 the function f is not required to be of exponential type. Since ϕ can be any non-decreasing, non-negative function, f may grow arbitrarily fast on parallels to the real line. However, when f is not of exponential type, then it is not guaranteed that the error bound will approach zero as $N \to \infty$.

We now turn to entire functions of exponential type. The directional growth of such a function is described by the indicator; see [2, Ch. 5]. From the properties of the indicator, it follows that every entire function f of exponential type can be estimated as

$$|f(\xi + \mathrm{i}\eta)| \le M\mathrm{e}^{\kappa|\xi|+\sigma|\eta|} \qquad (\xi, \eta \in \mathbb{R}), \tag{21}$$

where M, κ, and σ are non-negative numbers. Conversely, if an entire function f satisfies (21), then it is of exponential type $(\kappa^2 + \sigma^2)^{1/2}$.

For $\alpha > 0$ we shall say that an error bound converges to zero *exponentially of order* α as $N \to \infty$ if it has the form (6). In the following corollary it is remarkable that the error bound converges exponentially to zero while f may be *any* entire function of exponential type including those that *increase* exponentially on the real line. We do not know of *any other result* in sinc approximation where the samples are allowed to grow exponentially.

In this connection, it may be of interest to recall a conjecture [4, p. 620, Problem 16(II)] which states that for $\tau \in (0, \pi)$ it is not possible to have a formula

$$f(z) = \sum_{n=-\infty}^{\infty} f(n)\Lambda_{\tau,n}(z) \qquad (z \in \mathbb{C})$$

that holds for all $f \in \mathcal{E}_\tau$.

Corollary 2.2 *Let f be an entire function satisfying* (21) *with non-negative numbers M, κ, and σ, and let $h \in (0, \pi/(\sigma + 2\kappa))$. Then, in the notation of Theorem 2.1,*

$$\left|f(z) - \mathcal{C}_{h,N}[f](z)\right| \le \left|\sin(h^{-1}\pi z)\right| \frac{2\mathrm{e}^{-(\alpha-h\kappa)N}}{\sqrt{\pi\alpha N}} M\mathrm{e}^{\kappa(|\Re z|+h)} \beta_N(h^{-1}\Im z).$$

Proof Setting $\phi(x) = M\mathrm{e}^{\kappa x}$, we obtain the result as an immediate consequence of Theorem 2.1. □

It is not surprising that exponential growth of f on the real line, as it is admissible when κ in (21) is positive, affects the order of exponential convergence of the error bound. However, polynomial growth on the real line does *not* affect the order of exponential convergence.

Corollary 2.3 *Let f be an entire function of exponential type σ satisfying*

$$|f(\xi)| \le M\left(1 + \xi^2\right)^{k/2} \qquad (\xi \in \mathbb{R}), \tag{22}$$

where $M \geq 0$ and $k \in \mathbb{N}_0$, and let $h \in (0, \pi/\sigma) \cap (0, 1]$. Then, in the notation of Theorem 2.1,

$$\left|f(z) - \mathcal{C}_{h,N}[f](z)\right| \leq \left|\sin(h^{-1}\pi z)\right| \frac{2N^{k-1/2}\mathrm{e}^{-\alpha N}}{\sqrt{\pi\alpha}} MB_{k,N}(h^{-1}z)\beta_N(h^{-1}\Im z),$$

where

$$B_{k,N}(\zeta) := \left[\left(1 + \frac{1 + |\Re\zeta|}{N}\right)^2 + \left(1 + \frac{1 + |\Im\zeta|}{N}\right)^2\right]^{k/2}.$$

Proof First, suppose that $\sigma < \pi$ and $h = 1$. Consider

$$g(z) := \frac{f(z)}{(z + \mathrm{i})^k}. \tag{23}$$

Obviously, g is analytic and of exponential type σ in the closed upper half-plane. Moreover, $|g(\xi)| \leq M$ for $\xi \in \mathbb{R}$. Hence, by [2, Theorem 6.2.4], we have

$$|g(\xi + \mathrm{i}\eta)| \leq M\mathrm{e}^{\sigma\eta} \qquad (\xi,\, \eta \in \mathbb{R},\, \eta \geq 0),$$

and so

$$|f(\xi + \mathrm{i}\eta)| \leq M\left[\xi^2 + (|\eta| + 1)^2\right]^{k/2} \mathrm{e}^{\sigma|\eta|} \tag{24}$$

for $\xi \in \mathbb{R}$ and $\eta \geq 0$. A similar consideration for the lower half-plane with i replaced by $-\mathrm{i}$ in (23) shows that (24) holds for all $\xi, \eta \in \mathbb{R}$.

Next we note that in the proof of Theorem 2.1 the estimate (10) is needed for $|\eta| \leq N + |y|$ only. Under this restriction we see from (24) that (10) holds with

$$\phi(|\xi|) = M\left[\xi^2 + (|y| + N + 1)^2\right]^{k/2}.$$

Now the conclusion for $h = 1$ is an immediate consequence of Theorem 2.1.

When $0 < h < 1$, which is necessarily the case when $\sigma > \pi$, we consider $\widetilde{f}(z) := f(hz)$. Then $\widetilde{f}$ is of exponential type $h\sigma$, which is less than π. Furthermore,

$$\left|\widetilde{f}(\xi)\right| \leq M\left[1 + (h\xi)^2\right]^{k/2} \leq M\left(1 + \xi^2\right)^{k/2} \qquad (\xi \in \mathbb{R}). \tag{25}$$

This shows that the hypotheses of the corollary hold for $\widetilde{f}$. Now, the first part of the proof gives an error bound for $\widetilde{f}$ from which the desired result for f is deduced by replacing z by z/h. □

We note that f satisfies the hypotheses of Corollary 2.3 with $k = 0$ if and only if $f \in \mathcal{B}_\sigma^\infty$ and then M can be taken as $\|f\|_\infty$. In this case, the additional restriction that $h \leq 1$, which was needed in (25) only, can be dropped.

Corollary 2.4 *Let $f \in \mathcal{B}_\sigma^\infty$, where $\sigma > 0$. Then, in the notation of Theorem 2.1, we have*

$$\left|f(z) - \mathcal{C}_{h,N}[f](z)\right| \le \left|\sin(h^{-1}\pi z)\right| \frac{2\mathrm{e}^{-\alpha N}}{\sqrt{\pi\alpha N}} \|f\|_\infty \cdot \beta_N(h^{-1}\Im z). \qquad (26)$$

Remark If in addition to the hypotheses of Corollary 2.4 we have $f \in L^2(\mathbb{R})$, then $\|f\|_\infty \le \sqrt{\sigma/\pi}\|f\|_2$; see [9, p. 319]. This allows us to compare (9) with (26) for $\Im z = 0$.

In view of Corollary 2.4, we may ask for the best piecewise approximation of a function $f \in \mathcal{B}_\sigma^p$ by a linear combination of $2N+1$ successive samples $f(hn)$. Clearly, by scaling f appropriately, we can restrict ourselves to the case where $h = 1$. Furthermore, as far as piecewise approximation on the *real line* is concerned, it is enough to consider the interval $[-1/2, 1/2)$ since $f \in \mathcal{B}_\sigma^p$ implies that the translates $f(\cdot + k)$ also belong to $\mathcal{B}_\sigma^p$. Hence, a normalized form of the problem may be stated as follows.

Problem Let $p \in [1, +\infty]$ and $\sigma \in (0, \pi)$ be given. Let Λ be the collection of all sequences

$$\lambda := \{\lambda_{n,N} \,:\, n = 0, \pm 1, \dots, \pm N,\ N \in \mathbb{N}\}$$

of functions

$$\lambda_{n,N} \,:\, \left[-\frac{1}{2}, \frac{1}{2}\right) \longrightarrow \mathbb{C}.$$

For $f \in \mathcal{B}_\sigma^p$, define

$$E_N(f, \lambda, p, \sigma) := \sup_{-1/2 \le x < 1/2} \left| f(x) - \sum_{n=-N}^{N} f(n)\lambda_{n,N}(x) \right|,$$

$$E(f, \lambda, p, \sigma) := \limsup_{N\to\infty} \frac{\ln E_N(f, \lambda, p, \sigma)}{N},$$

$$E(\lambda, p, \sigma) := \sup\left\{E(f, \lambda, p, \sigma) \,:\, f \in \mathcal{B}_\sigma^p\right\},$$

and determine

$$E(p, \sigma) := \inf\left\{E(\lambda, p, \sigma) \,:\, \lambda \in \Lambda\right\}.$$

From Corollary 2.4, it follows that

$$-\infty \le E(p, \sigma) \le -\frac{\pi - \sigma}{2}$$

for any $p \in [1, +\infty]$. As a next step, it may be interesting to know whether

$$-\infty < E(p, \sigma) < -\frac{\pi - \sigma}{2}.$$

3 Functions Analytic in a Strip

For $d > 0$, we introduce the strip

$$\mathcal{S}_d := \left\{ z \in \mathbb{C} \,:\, |\Im z| < d \right\}.$$

First we recall a result for the classical sampling series. It is a version of [13, Theorem 3.1.7 for $s = \infty$].

Theorem C *Let f be analytic and bounded on the strip $\mathcal{S}_d$, and suppose that*

$$|f(x)| \le c\mathrm{e}^{-\lambda|x|} \qquad (x \in \mathbb{R}) \tag{27}$$

for some positive numbers c and λ. For $N \in \mathbb{N}$,

$$h := \left(\frac{\pi d}{\lambda N}\right)^{1/2},$$

and $z \in \mathcal{S}_d$, define

$$C_N[f](z) := \sum_{n=-N}^{N} f(hn)\mathrm{sinc}\,(h^{-1}z - n).$$

Then there exists a positive number c_1, depending only on f, d, λ, and $y := \Im z$, such that

$$|f(z) - C_N[f](z)| \le c_1\sqrt{N}\exp\left(-(d-|y|)\sqrt{\frac{\pi\lambda N}{d}}\right).$$

A result in the spirit of Theorem 2.1 is as follows.

Theorem 3.1 *Let f be analytic in $\mathcal{S}_d$ such that*

$$|f(\xi + \mathrm{i}\eta)| \le \phi(|\xi|) \qquad (\xi, \eta \in \mathbb{R},\ |\eta| < d),$$

where ϕ is a continuous, non-decreasing, non-negative function on $[0, \infty)$. For $N \in \mathbb{N}$, $h := d/N$, $z \in \mathcal{S}_{d/4}$, and $N_z := \lfloor \Re z + \frac{1}{2} \rfloor$, define

$$\mathcal{C}_N[f](z) := \sum_{n=N_{z/h}-N}^{N_{z/h}+N} f(hn)\,\mathrm{sinc}\left(\frac{z}{h} - n\right)\exp\left(-\frac{\pi}{2N}\left(\frac{z}{h} - n\right)^2\right). \tag{28}$$

Then

$$\begin{aligned} \left|f(z) - \mathcal{C}_N[f](z)\right| \;\le\; & \left|\sin(h^{-1}\pi z)\right| \frac{2\sqrt{2}}{\pi\sqrt{N}}\,\mathrm{e}^{-\pi N(1-2q)/2} \\ & \times \phi(|\Re z| + d + h)\cdot \gamma_N(q)\,, \end{aligned} \tag{29}$$

where $q := |\Im z|/d$ *and*

$$\begin{aligned}\gamma_N(q) &:= \frac{1}{1-q}\left[\frac{1}{1-\mathrm{e}^{-2\pi N}} + \frac{2\sqrt{2}}{\pi\sqrt{N}(1+q)}\right]\\ &= \frac{1}{1-q}\left[1+O(N^{-1/2})\right] \qquad (N\to\infty).\end{aligned}$$

Proof It is enough to prove the error estimate for functions which are analytic in the *closure* of $\mathcal{S}_d$. In fact, if this has been done and we have a function that satisfies nothing more than the hypotheses of Theorem 3.1, then (29) will hold with d replaced by $d-\varepsilon$ for sufficiently small positive ε. But in this result the two sides of the error estimate depend continuously on ε. Hence, the limit $\varepsilon \to 0+$ amounts to replacing ε by zero.

It is also enough to prove the theorem for an arbitrary but fixed $N \in \mathbb{N}$. For doing this, we may assume that $d = N$, and so $h = 1$, by scaling the argument of f appropriately. As explained in the proof of Theorem 2.1, we may also assume that $y := \Im z \geq 0$.

After these specializations we set $N' := N + \frac{1}{2}$, $\omega := \pi/(2N)$, and denote by $\mathcal{R}$ the positively oriented rectangle with vertices at $\pm N' + N_z - \mathrm{i}(N-y)$ and $\pm N' + N_z + \mathrm{i}N$. Using again the kernel (13), we have by the residue theorem

$$\begin{aligned}E &:= f(z) - \mathcal{C}_N[f](z) = \int_{\mathcal{R}} K(z,\zeta)\,\mathrm{d}\zeta\\ &= \frac{\sin\pi z}{2\pi\mathrm{i}}\left(I_{\text{hor}}^{-} + I_{\text{vert}}^{+} + I_{\text{hor}}^{+} + I_{\text{vert}}^{-}\right),\end{aligned} \tag{30}$$

where

$$I_{\text{hor}}^{+} = -\int_{-N'+N_z}^{N'+N_z} \frac{f(t+\mathrm{i}N)\,G(\sqrt{\omega}(z-t-\mathrm{i}N))}{(t-z+\mathrm{i}N)\sin\pi(t-\mathrm{i}N)}\,\mathrm{d}t\,,$$

$$I_{\text{hor}}^{-} = \int_{-N'+N_z}^{N'+N_z} \frac{f(t-\mathrm{i}(N-y))\,G(\sqrt{\omega}(x-t+\mathrm{i}N))}{(t-x-\mathrm{i}N)\sin\pi(t-\mathrm{i}(N-y))}\,\mathrm{d}t\,,$$

and

$$I_{\text{vert}}^{\pm} = \pm\mathrm{i}\int_{-N+y}^{N} \frac{f(\pm N' + N_z + \mathrm{i}t)\,G(\sqrt{\omega}(z \mp N' - N_z - \mathrm{i}t))}{(\pm N' + N_z + \mathrm{i}t - z)\sin\pi(\pm N' + N_z + \mathrm{i}t)}\,\mathrm{d}t\,.$$

Note that for every point $\zeta \in \mathcal{R}$ we have

$$|f(\zeta)| \leq \phi(N' + |N_z|) \leq \phi(|x| + N + 1).$$

Employing the inequalities (15)–(18), we easily find:

$$\left|I_{\mathrm{hor}}^{+}\right| \le \frac{2\phi(|x|+N+1)}{N-y}\cdot\frac{\mathrm{e}^{\omega(N-y)^2-\pi N}}{1-\mathrm{e}^{-2\pi N}}\sqrt{\frac{\pi}{\omega}}\,, \tag{31}$$

$$\left|I_{\mathrm{hor}}^{-}\right| \le \frac{2\phi(|x|+N+1)}{N}\cdot\frac{\mathrm{e}^{\omega N^2-\pi(N-y)}}{1-\mathrm{e}^{-2\pi(N-y)}}\sqrt{\frac{\pi}{\omega}}\,, \tag{32}$$

$$\left|I_{\mathrm{vert}}^{\pm}\right| \le \frac{2\phi(|x|+N+1)}{N}\cdot\mathrm{e}^{-\omega N^2}\int_{-N+y}^{N}\mathrm{e}^{\omega(y-t)^2-\pi|t|}\,\mathrm{d}t\,. \tag{33}$$

Clearly,

$$\omega(N-y)^2-\pi N \le \omega N^2-\pi(N-y) = -\frac{\pi}{2}(N-2y)\,.$$

It can also be verified that

$$(N-y)\left(1-\mathrm{e}^{-2\pi N}\right) \le N\left(1-\mathrm{e}^{-2\pi(N-y)}\right).$$

Hence, (31) and (32) can be simplified to

$$\left|I_{\mathrm{hor}}^{\pm}\right| \le \frac{2\phi(|x|+N+1)}{N-y}\cdot\frac{\mathrm{e}^{-\pi(N-2y)/2}}{1-\mathrm{e}^{-2\pi N}}\sqrt{\frac{\pi}{\omega}}\,. \tag{34}$$

In order to estimate the integral in (33), we replace the exponent of e by the following piecewise linear majorant:

$$\omega(y-t)^2-\pi|t| \le \begin{cases}\omega y^2+t(\pi/2-\omega y) & \text{if} \quad t\in[-N+y,0],\\ \omega y^2-t(\pi/2+2\omega y) & \text{if} \quad t\in[0,N].\end{cases}$$

This leads us to

$$\begin{aligned}\left|I_{\mathrm{vert}}^{\pm}\right| &< \frac{2\phi(|x|+N+1)}{N}\,\mathrm{e}^{-\omega(N^2-y^2)}\left[\frac{1}{\pi/2-\omega y}+\frac{1}{\pi/2+2\omega y}\right]\\ &\le \frac{2\phi(|x|+N+1)}{N}\,\mathrm{e}^{-\pi(N-2y)/2}\left[\frac{1}{\pi/2-\omega y}+\frac{1}{\pi/2+\omega y}\right]\\ &= \frac{8\phi(|x|+N+1)}{\pi N}\,\mathrm{e}^{-\pi(N-2y)/2}\,\frac{1}{1-(y/N)^2}\,.\end{aligned} \tag{35}$$

The proof is completed by combining (30), (34), and (35). □

Let us add a few comments on the previous results. In the method of contour integration along a rectangle, the contributions coming from the vertical line segments determine the truncation error while those coming from the horizontal line segments determine the aliasing error. Note that in Theorem B and in Theorem 2.1 the numbers N and h can be chosen independently while in Theorem C and in Theorem 3.1 they are correlated. The correlation is necessary in order to balance the two kinds of error.

We observe that the accuracy of the operator $\mathcal{C}_N[f]$ in (28) compares with that of $C_M[f]$ in Theorem C if λ in (27) is not smaller than $\pi/(4d)$ and M is about N^2. Moreover, in Theorem 3.1 no decay of $|f(x)|$ as $x \to \pm\infty$ is required. On the other hand, Theorem C has the advantage that its error bound converges to zero as $N \to \infty$ for each $z \in \mathcal{S}_d$ while in Theorem 3.1 this is only the case when $z \in \mathcal{S}_{d/4}$. Since for $z = x + \mathrm{i}y$ with $x, y \in \mathbb{R}$ we have

$$\sinh\left(\pi N \frac{|y|}{d}\right) \le \left|\sin\left(\pi \frac{z}{h}\right)\right| \le \cosh\left(\pi N \frac{y}{d}\right),$$

the decisive factor in the error bound (29) becomes

$$\exp\left(-\frac{\pi N}{2}\left(1 - \frac{4\,|y|}{d}\right)\right),$$

which guarantees convergence to zero for $|y| < d/4$ only. This restriction is genuine. It comes from the fact that $|G(x + \mathrm{i}y)|$ increases rapidly as $|y|$ increases.

In signal processing, a function $f \in L^2(\mathbb{R})$ is also called *a signal of finite energy*. It has a Fourier transform

$$\widehat{f}(u) := \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) \mathrm{e}^{-\mathrm{i}ux}\, \mathrm{d}x,$$

where the integral must be defined as a limit in the L^2 norm; see [5, Definition 2.12]. As a consequence of Theorem 3.1, we obtain a result for signals which are not necessarily bandlimited, that is, the support of their Fourier transform need not be bounded.

Corollary 3.2 *Let f be a continuous signal of finite energy such that*

$$\left|\widehat{f}(u)\right| \le M \mathrm{e}^{-\kappa|u|} \qquad (u \in \mathbb{R}) \tag{36}$$

for some positive numbers M and κ. For $N \in \mathbb{N}$ and $d \in (0, \kappa)$, set $h := d/N$. Then, for $x \in \mathbb{R}$, we have

$$\begin{aligned} |f(x) - \mathcal{C}_N[f](x)| &\le \left|\sin\left(h^{-1}\pi x\right)\right| \frac{4M}{\pi^{3/2}(\kappa - d)} \cdot \frac{\mathrm{e}^{-\pi N/2}}{\sqrt{N}}\, \gamma_N(0) \\ &\le \frac{1.37\, M}{\kappa - d} \cdot \frac{\mathrm{e}^{-\pi N/2}}{\sqrt{N}}. \end{aligned}$$

Proof For $z = x + \mathrm{i}y$, where $x, y \in \mathbb{R}$ and $|y| \le d$, we consider the function g defined by

$$g(z) := \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \widehat{f}(u) \mathrm{e}^{\mathrm{i}zu} \, \mathrm{d}u \,. \tag{37}$$

Using (36), we find that

$$\begin{aligned} |g(z)| &\le \frac{M}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \mathrm{e}^{-\kappa|u|-yu} \, \mathrm{d}u \\ &\le \frac{2M}{\sqrt{2\pi}} \int_{0}^{\infty} \mathrm{e}^{-u(\kappa-|y|)} \, \mathrm{d}u \\ &\le \sqrt{\frac{2}{\pi}} \cdot \frac{M}{\kappa - d} \,. \end{aligned}$$

From this we conclude that g is analytic in the strip $\mathcal{S}_d$ and satisfies the hypotheses of Theorem 3.1 with

$$\phi(\xi) \equiv \sqrt{\frac{2}{\pi}} \cdot \frac{M}{\kappa - d} \,.$$

Furthermore, $f(x) = g(x)$ for $x \in \mathbb{R}$ as a consequence of the Fourier inversion theorem and the continuity of f. Now, applying Theorem 3.1 to g and substituting $z = x \in \mathbb{R}$, we obtain the desired result. □

Under the hypotheses of Corollary 3.2, an error bound for the approximation by the truncated classical sampling series can be deduced from a result by Butzer and Stens [3, Theorem 1].

4 Examples

We restrict ourselves to examples which are not accessible by the results in [8]–[12].

Example 1 Let $f(z) = \cos z$. In this case Corollary 2.4 applies with $\sigma = 1$ and $\|f\|_\infty = 1$. Since $\mathcal{C}_{h,N}[f](x)$ duplicates $f(x)$ at the points jh for $j \in \mathbb{Z}$, it seems interesting to consider the absolute errors at the intermediate points $x_{h,j} := (j - \frac{1}{2})h$. Numerical results are given in Tables 1 to 3. As predicted by the theory, the number of correct digits roughly doubles when N doubles. Furthermore, without any additional cost, the precision increases when N is fixed but h decreases. However, decreasing h means that the samples are taken from a denser set. Note that in Tables 1 to 3 corresponding lines contain results for comparable points $x_{h,j}$. For example, $x_{1,11} = 10.5$ and $x_{1/4,41} = 10.125$. The

$h=1$	abs. errors for $N=5$		abs. errors for $N=10$		abs. errors for $N=20$	
j	true value	bound	true value	bound	true value	bound
1	9.21E-04	3.43E-03	3.04E-06	1.04E-05	5.38E-11	1.52E-10
3	7.76E-04	3.43E-03	2.98E-06	1.04E-05	4.64E-11	1.52E-10
5	2.75E-04	3.43E-03	5.59E-07	1.04E-05	1.52E-11	1.52E-10
7	1.01E-03	3.43E-03	3.45E-06	1.04E-05	5.90E-11	1.52E-10
9	5.61E-04	3.43E-03	2.31E-06	1.04E-05	3.40E-11	1.52E-10
11	5.38E-04	3.43E-03	1.52E-06	1.04E-05	3.08E-11	1.52E-10

Table 1: Approximation of cos at $x_{h,j}$ for $h = 1$.

$h=\frac{1}{2}$	abs. errors for $N=5$		abs. errors for $N=10$		abs. errors for $N=20$	
j	true value	bound	true value	bound	true value	bound
1	2.85E-04	8.56E-04	2.70E-07	7.47E-07	3.41E-13	9.03E-13
5	1.92E-04	8.56E-04	1.86E-07	7.47E-07	2.31E-13	9.03E-13
9	1.25E-04	8.56E-04	1.15E-07	7.47E-07	1.49E-13	9.03E-13
13	2.96E-04	8.56E-04	2.82E-07	7.47E-07	3.55E-13	9.03E-13
17	1.21E-04	8.56E-04	1.20E-07	7.47E-07	1.46E-13	9.03E-13
21	1.95E-04	8.56E-04	1.82E-07	7.47E-07	2.33E-13	9.03E-13

Table 2: Approximation of cos at $x_{h,j}$ for $h = \frac{1}{2}$.

$h=\frac{1}{4}$	abs. errors for $N=5$		abs. errors for $N=10$		abs. errors for $N=20$	
j	true value	bound	true value	bound	true value	bound
1	1.47E-04	4.32E-04	7.26E-08	2.02E-07	2.86E-14	7.02E-14
9	8.65E-05	4.32E-04	3.69E-08	2.02E-07	1.60E-14	7.02E-14
17	7.55E-05	4.32E-04	4.19E-08	2.02E-07	1.53E-14	7.02E-14
25	1.49E-04	4.32E-04	7.18E-08	2.02E-07	2.87E-14	7.02E-14
33	4.88E-05	4.32E-04	1.78E-08	2.02E-07	8.61E-15	7.02E-14
41	1.09E-04	4.32E-04	5.70E-08	2.02E-07	2.15E-14	7.02E-14

Table 3: Approximation of cos at $x_{h,j}$ for $h = \frac{1}{4}$.

error bounds, which do not depend on $x_{h,j}$, are quite realistic, that is, they do not overestimate the true error very much.

Figures 1–3 show the graphs of the error on the interval $[0, 10]$ for $N = 10$ and h equal 1, 1/2, and 1/4, respectively.

Example 2 Let $f(z) = \cosh z$. In this case Corollary 2.2 applies with $\kappa = 1$, $\sigma = 0$, and $M = 1$. Since $\cosh z = \cos(iz)$, we have essentially the same function as in Example 1, except that now the samples are taken on a line of maximal growth. This time, we restricted ourselves to $h = 1/4$ and computed the errors at points $x_{1/4,j} = (2j - 1)/8$ spreading over an interval of length 10,

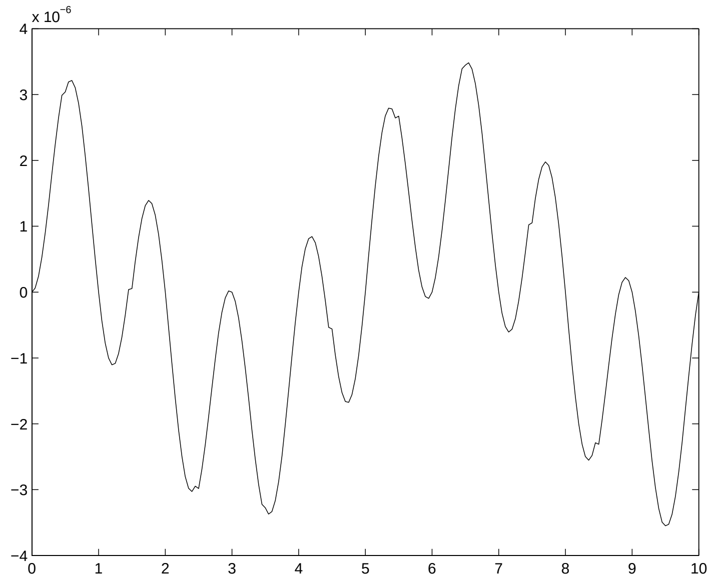

Figure 1: Graph of $\cos - \mathcal{C}_{h,N}[\cos]$ for $h = 1$, $N = 10$.

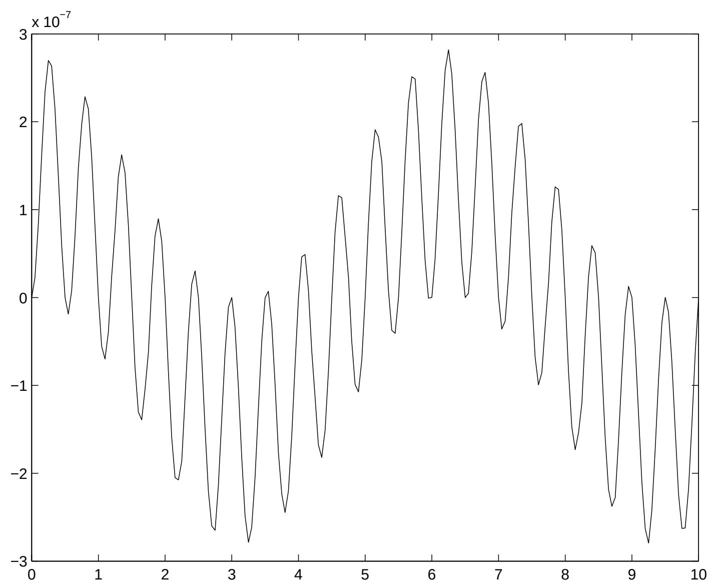

Figure 2: Graph of $\cos - \mathcal{C}_{h,N}[\cos]$ for $h = 1/2$, $N = 10$.

the largest point being $x_{1/4,41} = 10.125$. Numerical results are given in Tables 4 to 6. The absolute errors increase with j, but the relative errors are nearly constant for fixed N, and they have about the same size as the absolute errors in the previous example or are even a little smaller. Unfortunately, the error

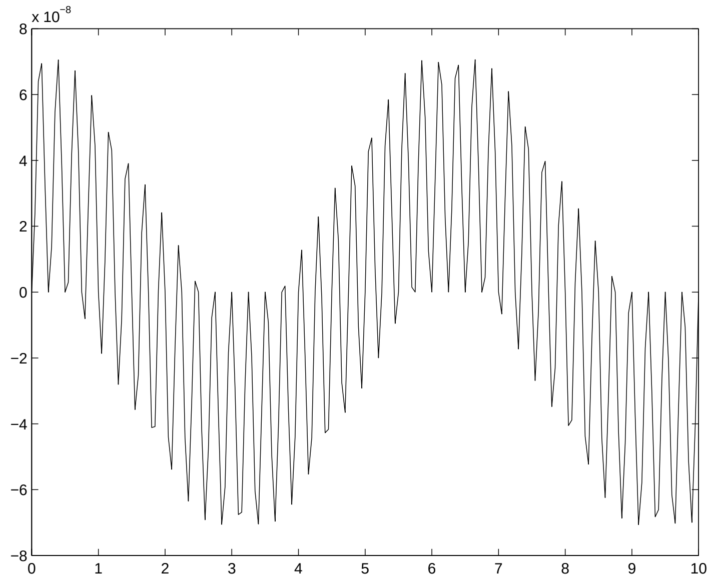

Figure 3: Graph of $\cos - \mathcal{C}_{h,N}[\cos]$ for $h = 1/4$, $N = 10$.

bounds overestimate the true errors quite a bit. Perhaps the estimates in the proof of Theorem 2.1 should be refined in the case where ϕ is not a constant.

Figures 4 and 5 show the graphs of the error $\cosh - \mathcal{C}_{h,N}[\cosh]$ and the signed relative error $(\cosh - \mathcal{C}_{h,N}[\cosh])/\cosh$ on the interval $[0, 10]$ for $h = 1/4$ and $N = 5$. The calculations are quite sensitive to round-off errors. This phenomenon is a price we must pay for approximating a function from exponentially increasing samples. It comes from the fact that such a small quantity like the true error is calculated via large numbers, such as $\cosh 10$. For the tables a higher precision has been used than for the graphs. This may explain why the values in the fourth column of Table 4 differ from the corresponding values in Fig. 5 by about 15%.

$h = \frac{1}{4}$	abs. errors for $N = 5$		rel. errors for $N = 5$	
j	true value	bound	true value	bound
1	3.08E-05	1.11E-03	3.06E-05	1.10E-03
9	1.59E-04	8.23E-03	3.73E-05	1.94E-03
17	1.16E-03	6.08E-02	3.76E-05	1.96E-03
25	8.59E-03	4.49E-01	3.76E-05	1.96E-03
33	6.35E-02	3.32E 00	3.76E-05	1.97E-03
41	4.69E-01	2.45E 01	3.76E-05	1.97E-03

Table 4: Approximation of cosh at $x_{h,j}$ with $N = 5$.

$h = \frac{1}{4}$	abs. errors for $N = 10$		rel. errors for $N = 10$	
j	true value	bound	true value	bound
1	3.39E-08	9.77E-07	3.36E-08	9.69E-07
9	1.65E-07	7.22E-06	3.89E-08	1.70E-06
17	1.21E-06	5.33E-05	3.91E-08	1.72E-06
25	8.94E-06	3.94E-04	3.91E-08	1.72E-06
33	6.60E-05	2.91E-03	3.91E-08	1.72E-06
41	4.88E-04	2.15E-02	3.91E-08	1.72E-06

Table 5: Approximation of cosh at $x_{h,j}$ with $N = 10$.

$h = \frac{1}{4}$	abs. errors for $N = 20$		rel. errors for $N = 20$	
j	true value	bound	true value	bound
1	6.77E-15	1.19E-12	6.72E-15	1.18E-12
9	6.79E-15	8.76E-12	1.60E-15	2.06E-12
17	4.43E-14	6.48E-11	1.43E-15	2.09E-12
25	3.26E-13	4.78E-10	1.43E-15	2.09E-12
33	2.41E-12	3.54E-09	1.43E-15	2.09E-12
41	1.78E-11	2.61E-08	1.43E-15	2.09E-12

Table 6: Approximation of cosh at $x_{h,j}$ with $N = 20$.

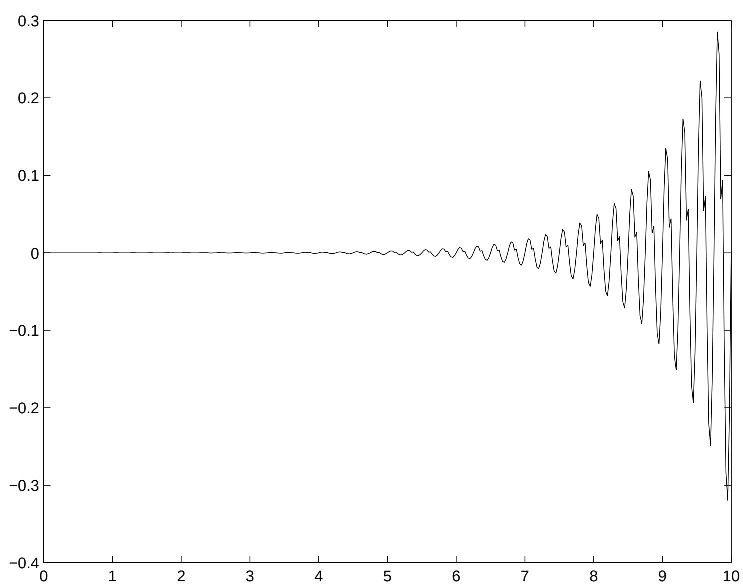

Figure 4: Graph of $\cosh - \mathcal{C}_{h,N}[\cosh]$ for $h = 1/4$, $N = 5$.

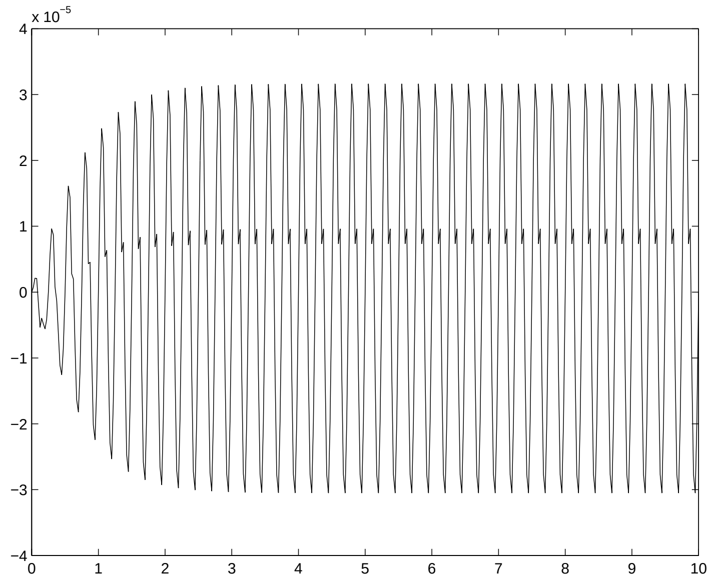

Figure 5: Graph of $(\cosh - \mathcal{C}_{h,N}[\cosh])/\cosh$ for $h = 1/4$, $N = 5$.

Example 3 Let $f(z) = (1 + z^2)^{1/2}$, taking that branch of the root which is positive on the real line. In this case Theorem 3.1 applies with $d = 1$ and

$$\phi(x) = \begin{cases} (1 + x^2)^{1/2} & \text{for} \quad x^2 \le \dfrac{1}{2}\,, \\ (1 + 4x^2 + x^4)^{1/4} & \text{for} \quad x^2 \ge \dfrac{1}{2}\,. \end{cases}$$

First we computed the absolute errors and the error bounds of Theorem 3.1 for an approximation by $\mathcal{C}_N[f]$ at points $t_{N,j} := (2j-1)/(2N)$. The results are shown in Table 7. The table has been arranged in such a way that all results located in one line belong to comparable points $t_{N,j}$. It is seen that the errors grow very slowly for fixed N and increasing j. The error bounds are very realistic. Graphs of the error on the interval $[0, 5]$ for $N = 5$ are shown in Figures 6 and 7.

In a second test, we considered the quality of approximation at the non-real points iy; see Table 8. As predicted by the theory, we have exponential convergence when $|y| < 1/4$ while the errors increase with an exponential rate when $|y| > 1/4$.

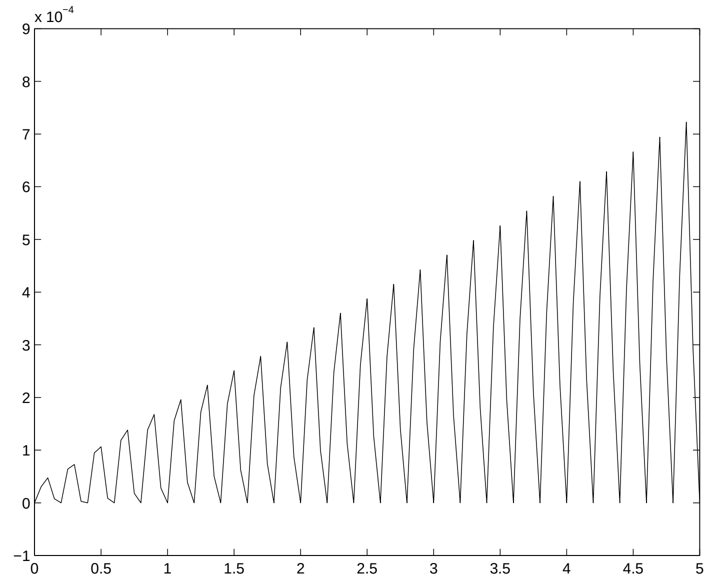

Figure 6: Graph of the error for Example 3 for $N = 5$.

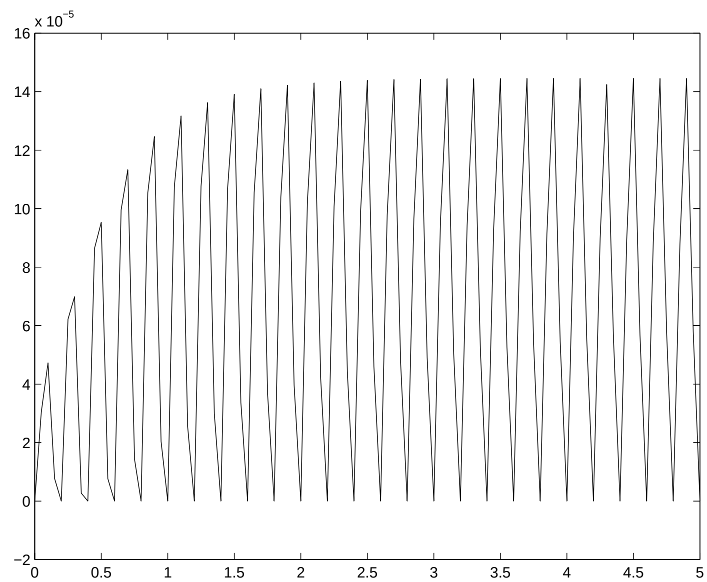

Figure 7: Graph of the relative error for Example 3 for $N = 5$.

Absolute errors for $N=5$			Absolute errors for $N=15$			Absolute errors for $N=25$		
j	true value	bound	j	true value	bound	j	true value	bound
5	1.68E-04	5.45E-04	15	1.63E-11	4.07E-11	25	1.94E-18	4.52E-18
10	3.05E-04	7.43E-04	30	2.86E-11	5.58E-11	50	3.39E-18	6.21E-18
15	4.43E-04	9.45E-04	45	4.10E-11	7.15E-11	75	4.86E-18	7.97E-18
20	5.82E-04	1.16E-03	60	5.37E-11	8.75E-11	100	6.36E-18	9.76E-18
25	7.23E-04	1.14E-03	75	6.66E-11	1.04E-10	125	7.88E-18	1.16E-17

Table 7: Approximation of $(1+z^2)^{1/2}$ at $t_{N,j} := \frac{2j-1}{2N}$.

Absolute errors for $N=5$			Absolute errors for $N=15$			Absolute errors for $N=25$		
y	true value	bound	y	true value	bound	y	true value	bound
0.05	5.11E-05	7.48E-04	0.05	1.40E-10	1.56E-09	0.05	3.72E-16	4.02E-15
0.10	3.90E-04	4.53E-03	0.10	1.95E-08	1.83E-07	0.10	1.21E-12	1.09E-11
0.15	2.31E-03	2.36E-02	0.15	2.52E-06	2.15E-05	0.15	3.63E-09	2.95E-08
0.20	1.24E-02	1.20E-01	0.20	3.14E-04	2.52E-03	0.20	1.05E-05	8.02E-05
0.25	6.48E-02	6.12E-01	0.25	3.82E-02	2.97E-01	0.25	2.97E-02	2.19E-01
0.30	3.34E-01	3.12E 00	0.30	4.57E 00	3.52E 01	0.30	8.23E 01	6.02E 02
0.35	1.70E 00	1.61E 01	0.35	5.40E 02	4.20E 03	0.35	2.25E 05	1.66E 06

Table 8: Approximation of $(1+z^2)^{1/2}$ at non-real points iy.

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