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Modified Wilson Orthonormal Bases

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Abstract

Wilson bases are constituted by trigonometric functions multiplied by translates of a window function that, in turn, is an atom of Gabor tight frame. We study the Wilson systems with non-classical sign sequences obtaining the characterization of the atoms for which these non-classical Wilson systems are orthonormal bases. The real-valued functions satisfy our characterization condition for the classical case. The operator intertwining the members of the Wilson system formula is constructed explicitly.

Key words and phrases : Hilbert spaces, Gabor systems, Wilson bases, frame operator, tight frames, modified Wilson system, intertwining operator

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1 Introduction

Gabor systems are time– and frequency–shifted images of a single function f, called atom. Contrary to wavelets, they are known not to be orthonormal bases as long as atom f has good localization properties (Balian-Low Theorem cf. [2] and [15] and its recent extensions, e.g., [3]). However, replacing the frequency-shift (modulation) with multiplication by suitably chosen trigonometric functions yields a Wilson system that under certain conditions *is* an orthonormal basis.

Wilson orthonormal bases were constructed by I. Daubechies et al. in [7]. The unconditionality of Wilson systems in the class of coorbit spaces was analyzed in [8] leading to the construction of Riesz basis for Bargmann space series in [10]. Generalizations of Wilson bases to non-rectangular lattices are discussed in [14]. Approximation properties of Wilson bases depend on polynomials they can reproduce [4]. The applications of cosine-modulated filter banks were dealt with in [5], [6]. These filter banks are also based on Wilson bases.

In [7], where the construction was given, the following theorem (cf. Proposition 5.2 [7]) is proved, which we quote in a somehow restated version and apply not to the original system but rather to its Fourier transform counterpart. So let M and T be unitary operators in $L^2(\mathbb{R})$ defined by

$$Mh(x) = e^{\pi i x}h(x), \quad Th(x) = h(x-1) \quad \text{for any} \quad h \in L^2(\mathbb{R}).$$

The operators, called modulation and translation, respectively, implement frequency- and time-shifts on functions from $L^2(\mathbb{R})$. Then one has the following.(The extension starting from *"Under the same assumptions..."* is due to P. Auscher [1].)

Theorem 1.1 (Daubechies-Jaffard-Journé 1991, Auscher 1994)
If $(M^m T^n f)_{m,n\in\mathbb{Z}}$ is a tight frame in $L^2(\mathbb{R})$, $\|f\| = 1$, and f is real-valued, then the system composed of $(M^{2m} f)_{m\in\mathbb{Z}}$ and

$$\Big[2^{-1/2}(M^m T^n f + (-1)^{m+n} M^m T^{-n} f)\Big]_{n\geq 1,\, m\in\mathbb{Z}} \tag{1}$$

is an orthonormal basis in $L^2(\mathbb{R})$. Under the same assumptions, the system composed of $(M^{2m+1} f)_{m\in\mathbb{Z}}$ and

$$\Big[2^{-1/2}(M^m T^n f - (-1)^{m+n} M^m T^{-n} f)\Big]_{n\geq 1,\, m\in\mathbb{Z}} \tag{2}$$

is also an orthonormal basis in $L^2(\mathbb{R})$.

The Fourier transform of the system described above contains the factor $(T^n f + (-1)^{m+n} T^{-n} f)\,\widehat{}\,(\xi) = \cos(2\pi n\xi)\widehat{f}(\xi)$ or $\sin(2\pi n\xi)\widehat{f}(\xi)$ depending on the parity of $m+n$. Therefore, the Wilson bases have the alternative name of *local trigonometric bases.*

In the sequel, we shall refer to the systems like (2) as *complementary* to (1). The difference is that the sign preceding the second member of Wilson system formula is different in these cases *but* independent on m, n. The term *complementary* also refers to the fact that from the original Wilson system and its complementary one can retrieve the original tight frame by a suitable linear combination of vectors.

If the sign independent of m, n is the same, but the signs before the second members of the formula differ between two Wilson systems by the factor $(-1)^m$ or $(-1)^n$, or both, we call such systems *modified* with respect to each other. In this situation the formula for the second member of the Wilson system involves a different inner automorphism of the underlying discrete group and, certainly, the unitary implementation operator also exists, if it existed for the *'non-modified'* case. The approach presented here applies as the interested reader will note the mentioned link between modified cases presented in this paper himself. To keep the paper short, clear, and focused we purposefully omitted this and similar connections towards the group and representation theories; so the reader interested in such is encouraged to consult [18] or [19] for the more detailed account on the topic.

Denote by $\mathcal{S}$ the set of all vectors $f \in L^2(\mathbb{R})$, $\|f\| = 1$ such that $(M^m T^n f)_{m,\,n\in\mathbb{Z}}$ is a tight frame. For $f \in \mathcal{S}$ it follows that the bound of this tight frame equals 2. Denote further by $\mathcal{W}$ the set of all vectors $f \in \mathcal{S}$ such that the Wilson system for f is an orthonormal basis. The objective of this paper is to provide a description of $\mathcal{W}$, where 'Wilson system' stands for the classical system (1) as well as for its modifications listed further in the formulae (5), (22), (23). In [1] the characterization of $\mathcal{W}$ is given for the classical Wilson system and in the present paper we extend P. Auscher's approach to embrace the non-classical sign sequences.

Despite similarity of results in these two approaches, P. Auscher's approach (and proof) involves properties of the appropriate Gabor operators when summed up over all $m, n \in \mathbb{Z}$, while we use the commutation property intertwining the single operators in both members of the Wilson system formula. And (unfortunately) there is no easy way to derive his characterization from ours.

Let us stress that we study the Fourier counterpart of the classical Wilson system investigated in the literature. We bear in mind that it may differ uncomfortably, but all results about being a tight frame with the given bound or an orthonormal basis are preserved by the Fourier transform as a unitary mapping. In our approach we explicitly define the operator intertwining the families of operators, for instance, $(M^m T^n f)_{m,n\in\mathbb{Z}}$ and $((-1)^m M^m T^{-n} f)_{m,n\in\mathbb{Z}}$. This type of intertwining property has been already analyzed for the classical Wilson system in [9, p. 170-171]. The approach presented here, however, does not give an answer to the question about whether or not there exist smooth, rapidly decaying atoms satisfying the characterization conditions for non-classical Wilson systems.

We start Section 2 with the necessary definitions and Theorem 2.1 characterizing $\mathcal{W}$ for the Wilson system with the sequence $(-1)^m$ replacing the sign sequence $(-1)^{m+n}$ in (1). Section 3 contains the brief presentation of the Zak transform and the variant of the characterization obtained by this tool. In Section 4 the approach from Sections 2 and 3 is applied to the classical case to demonstrate that the real-valued functions automatically fall into the set $\mathcal{W}$ as asserted in Theorem 1.1 and to obtain the geometric characterization of atoms from $\mathcal{W}$ obtained in [17, Theorem 25]. Then in Section 5 we describe how to modify the proof of Theorem 2.1 to cover the remaining cases of $(-1)^n$ and the constant sequence 1. Along with the discussed cases we also develop the characterization whose form is similar to Auscher's in [1] and the characterizations using the Zak transform. Section 6 contains the extension of the paper's results to the complementary Wilson systems that are modifications of (2).

2 The Main result

Let C be positive. A sequence of vectors $(x_n)_{n\in I} \subset \mathcal{H}$ for a countable set I, where $\mathcal{H}$ is a separable Hilbert space, is a tight frame with the bound C if for all $x \in \mathcal{H}$

$$\sum_{n\in I} |\langle x, x_n\rangle|^2 = C\|x\|^2_{\mathcal{H}}. \tag{3}$$

Fix the sequence of vectors $(x_n)_{n\in I}$ such that $\sum_{n\in I} |\langle x, x_n\rangle|^2 \le C\|x\|^2$; the frame operator for this sequence is defined by

$$Sx = \sum_{n\in I} \langle x, x_n\rangle\, x_n \tag{4}$$

for all $x \in \mathcal{H}$. Let J be a unitary operator in $L^2(\mathbb{R})$ defined by

$$Jh(x) = h(x - 2[x] + 1)$$

for any $h \in L^2(\mathbb{R})$, where $[x]$ is the largest integer smaller than x. In the sequel we consider the frame operator of the system $(M^mT^nf)_{m,n\in\mathbb{Z}}$ and will denote it by S. The frame operator of the Wilson system defined as the system composed of $(M^{2m}f)_{m\in\mathbb{Z}}$ and

$$\left[2^{-1/2}(M^mT^nf + (-1)^m M^mT^{-n}f)\right]_{n\ge 1,\, m\in\mathbb{Z}} \tag{5}$$

is denoted by W. Recall that $\mathcal{S}$ is the set of all such unit norm $f \in L^2(\mathbb{R})$ that $(M^mT^nf)_{m,n\in\mathbb{Z}}$ is a tight frame.

Theorem 2.1 *Let $f \in \mathcal{S}$. Then the Wilson system (5) is an orthonormal basis in $L^2(\mathbb{R})$ if and only if*

$$\forall_{m,n\in\mathbb{Z}} \quad \left\langle Jf, M^{2m}T^{2n}f\right\rangle = 0 \tag{6}$$

for all $m, n \in \mathbb{Z}$.

In the proof of Theorem 2.1 we shall use the following.

Lemma 2.2 *The operator J is of order 2 and satisfies*

$$JM^mT^n = (-M)^mT^{-n}J. \tag{7}$$

Proof of Lemma. Both operators T and J act by transforming the function domain; so let us introduce two mappings $t, j : \mathbb{R} \to \mathbb{R}$, namely, $tx = x - 1$, $jx = x - 2[x] + 1$. Fix now $x \in \mathbb{R}$. Then $t^{-1}jx = t^{-1}(x - 2[x] + 1) = x - 2[x] + 2$. Also $jtx = j(x-1) = x - 1 - 2[x-1] + 1 = x - 2[x] + 2$. Hence, $t^{-1}j = jt$ and $T^{-1}J = JT$. On the other hand,

$$JMf(x) = e^{\pi i(x-2[x]+1)}f(jx) = -e^{\pi ix}f(jx) = -MJf(x).$$

So, $JM = -MJ$ and $JM^mT^n = (-M)^mT^{-n}J$. $\square$

Note that in the above definitions of the operator J and the mapping j we could use the formulae with any odd integer n,

$$jx = x - 2[x] + n, \qquad Jf(x) = f(jx) = f(x - 2[x] + n).$$

For convenience we stick to the above choice of $n = 1$.

Proof of Theorem 2.1. Let us prove first that the Wilson system (5) is a tight frame with bound 1. Reasoning as in Lemma 5.3 [1] (cf. also Lemma 8.5.2, [9]), it is equivalent to

$$2W - S = \sum \left\langle \,\cdot\,, (-M)^m T^{-n} f \right\rangle M^m T^n f = 0, \tag{8}$$

since $S = 2\mathrm{Id}$. Indeed,

$$\begin{aligned} 2Wh \;&= \sum_{m\in\mathbb{Z}} \sum_{n\geq 1} \left\langle h, M^m(T^n f + (-1)^m T^{-n} f)\right\rangle M^m(T^n f + (-1)^m T^{-n} f) \\ &\qquad\qquad + 2\sum_{m\in\mathbb{Z}} \left\langle h, M^{2m} f\right\rangle M^{2m} f = \\ = \;& \sum_{m\in\mathbb{Z}} \sum_{n\neq 0} \langle h, M^m T^n f\rangle \; M^m T^n f + \sum_{m\in\mathbb{Z}} \langle h, M^m f\rangle\, M^m f + \\ + \;& \sum_{m\in\mathbb{Z}} \sum_{n\neq 0} \langle h, (-1)^m M^m T^{-n} f)\rangle \; M^m T^n f + \sum_{m\in\mathbb{Z}} \langle h, (-1)^m M^m f\rangle\, M^m f = \\ = \;& Sh + \sum_{m,n\in\mathbb{Z}} \left\langle h, (-1)^m M^m T^{-n} f)\right\rangle \; M^m T^n f. \end{aligned}$$

See also the detailed discussion of this type operator R in [9], p. 170-171.

Using Janssen's theorem ([13], sec. 1.4.1), we obtain the polarization

$$\sum_{m,n\in\mathbb{Z}} \langle \,\cdot\,, M^m T^n f_1\rangle \; M^m T^n f_2 \;=\; 2 \sum_{m,n\in\mathbb{Z}} \left\langle f_2, M^{2m} T^{2n} f_1\right\rangle \; M^{2m} T^{2n}. \tag{9}$$

We can relax the requirement of condition (A) (cf. [13]), since to assure the convergence of operators in the weak sense it suffices to consider the vectors in the dense subspaces of $L^2(\mathbb{R})$ where the condition (A) holds.

Inserting (7) into (8), we have by the polarization (9)

$$\begin{aligned} 2W - S \;&=\; \sum \left\langle \,\cdot\,, (-M)^m T^{-n} f\right\rangle \; M^m T^n f = \\ &=\; \sum \langle \,\cdot\,, J M^m T^n J f\rangle \; M^m T^n f = \\ &=\; \sum \langle J\,\cdot\,, M^m T^n J f\rangle \; M^m T^n f = \\ &=\; 2 \left[\sum \langle f, M^{2m} T^{2n} J f\rangle \; M^{2m} T^{2n} \right] J. \end{aligned}$$

So condition (8) holds if and only if $\left[\sum \left\langle f, M^{2m}T^{2n}Jf\right\rangle M^{2m}T^{2n}\right] = 0$. Using Janssen's representation [13, sec. 1.4.3], we immediately get that it is equivalent to

$$\forall_{m,n\in\mathbb{Z}} \quad \left\langle Jf, M^{2m}T^{2n}f\right\rangle = 0. \tag{10}$$

The orthonormality follows from the fact that the Wilson system vectors are of unit length. Certainly, $\|M^{2m}f\| = 1$. Also,

$$\|M^mT^nf + (-1)^m M^mT^{-n}f\|^2 = 2 + (-1)^m\ 2\Re\ \left\langle T^{2n}f, f\right\rangle,$$

where $\Re z$ stands for a real part of the complex number z. From the Wexler-Raz Identity [16] $\left\langle T^{2n}f, f\right\rangle = 0$ for $n \geq 1$, because $(M^mT^nf)_{m,n\in\mathbb{Z}}$ is a tight frame with bound 2. So

$$2^{-1/2}\left(M^mT^nf + (-1)^m M^mT^{-n}f\right)$$

is of unit length. □

The below characterization is similar to the characterization by P. Auscher in [1, Eqn. (5.5b)].

Proposition 2.3 *The condition (6) is equivalent to*

$$E_n(x) = \sum_k\ f(x-k-1)\ \overline{f(x-2n+k)}\ =\ 0 \tag{11}$$

for almost all $x \in [0,1]$ and for all $n \in \mathbb{Z}$.

Proof of Proposition 2.3. Indeed,

$$\begin{aligned} \left\langle Jf, M^{2m}T^{2n}f\right\rangle &= \int_{\mathbb{R}}[Jf](x)\ \overline{e^{2\pi imx}f(x-2n)}\ dx = \\ &= \int_{\mathbb{R}} f(x-2[x]-1)\ \overline{e^{2\pi imx}f(x-2n)}\ dx = \\ &= \int_{[0,1]}\sum_k f(y-k-1)\ \overline{e^{2\pi imy}f(y-2n+k)}\ dy = \\ &= \int_{[0,1]}\sum_k f(y-k-1)\ \overline{f(y-2n+k)}\ e^{-2\pi imy}\ dy. \end{aligned}$$

Thus, (6) holds if and only if

$$E_n(y) = \sum_k\ f(y-k-1)\ \overline{f(y-2n+k)}\ =\ 0 \tag{12}$$

for almost all $y \in [0,1]$ and for any $n \in \mathbb{Z}$. □

3 The Zak Transform Approach

To obtain yet another version of this characterization we will use the Zak transform. Let $Z : L^2(\mathbb{R}) \to L^2([0,1)^2)$ be the Zak transform with parameter 2 (for the detailed discussion of properties and applications of Zak transform see, for instance, [11] - [13], [20]) defined as:

$$Zf(t,\omega) = 2^{1/2} \sum_{n\in\mathbb{Z}} f(2(t-n))e^{2\pi in\omega}$$

with the following quasi-periodicity properties:

$$Zf(t+1,\omega) = e^{2\pi i\omega} Zf(t,\omega), \qquad Zf(t,\omega+1) = Zf(t,\omega).$$

Zak transform has the following properties for operators M^2 and T^2:

$$Z[M^2 f](t,\omega) = e^{4\pi it} Zf(t,\omega), \qquad Z[T^2 f](t,\omega) = e^{-2\pi i\omega} Zf(t,\omega).$$

Applying this to the both factors of the inner product in the characterization condition (6), we obtain by the unitarity of the Zak transform that (6) is equivalent to

$$\forall_{m,n\in\mathbb{Z}} \qquad \int_{[0,1]^2} Z\left[Jf\right](t,\omega)\ \overline{e^{4\pi imt}\ e^{-2\pi in\omega}\ Zf(t,\omega)}\, dt\, d\omega \ = \ 0$$

or

$$Z\left[Jf\right](t,\omega)\ \overline{Zf(t,\omega)} + Z\left[Jf\right](t+\tfrac{1}{2},\omega)\ \overline{Zf(t+\tfrac{1}{2},\omega)} \ = \ 0$$

for almost all $(t,\omega) \in \left[0,\frac{1}{2}\right) \times \left[0,1\right)$. One directly verifies then that

$$Z\left[Jf\right](t,\omega) \ = \ Zf(\zeta t, 1-\omega),$$

for $t,\omega \in [0,1]$, where $\zeta t = t - [2t] + \frac{1}{2}$, and that ζ cycles the points t and $t+\frac{1}{2}$ for any $t \in \left[0,\frac{1}{2}\right]$. Summarizing,

Proposition 3.1 *The condition (6) is equivalent to*

$$Zf(t,\omega)\ \overline{Zf(t+\tfrac{1}{2},1-\omega)} \ + \ Zf(t+\tfrac{1}{2},\omega)\ \overline{Zf(t,1-\omega)} \ = \ 0 \tag{13}$$

for almost all $\left(t,\omega\right) \in \left[0,\frac{1}{2}\right) \times \left[0,1\right)$.

4 The Classical Case

The classical Wilson system (1) is modified with respect to (5). Below we show a small modification of the proof of Theorem 2.1 leading to the characterization result in this case.

Theorem 4.1 *Let $f \in \mathcal{S}$. Then the Wilson system (1) is an orthonormal basis in $L^2(\mathbb{R})$ if and only if*

$$\forall_{m,n\in\mathbb{Z}} \quad \left\langle MJf, M^{2m}T^{2n}f \right\rangle = 0. \tag{14}$$

Proof of Theorem 4.1. Replacing J with MJ, we obtain an analogue of identity (7), namely,

$$(MJ)M^mT^n = (-M)^m(-T)^{-n}(MJ). \tag{15}$$

The appropriate characterization formula takes the form

$$\forall_{m,n\in\mathbb{Z}} \quad \left\langle MJf, M^{2m}T^{2n}f \right\rangle = 0. \tag{16}$$

□

P. Auscher's approach (and proof) involves properties of the appropriate one-dimensional projections onto the atom's images under the Gabor operators when summed over all $m, n \in \mathbb{Z}$ and yields – for the Fourier counterpart of the Wilson system we are using – the formula involving the inversion mapping $if(x) = f(-x)$. Since we work on other side of the Fourier transform we do not have this *inversion* effect.

Proposition 4.2 *The condition (14) is equivalent to*

$$E_n(x) = \sum_k (-1)^k\, f(x-k-1)\, \overline{f(x-2n+k)} \;=\; 0 \tag{17}$$

for almost all $x \in [0,1]$ and for any $n \in \mathbb{Z}$.

Slightly developing the argument from Proposition 3.1 we can obtain the following characterization of the classical case which also immediately demonstrates that in this case the real-valued atoms yield Wilson orthonormal bases. Indeed, for such atoms condition (18) is *par force* satisfied.

Proposition 4.3 *(cf. Theorem 25, [17]) The condition (14) is equivalent to*

$$Z\left[\Im f\right]\left(t,\omega\right)\, Z\left[\Re f\right]\left(t+\tfrac{1}{2},\omega\right) \;=\; Z\left[\Re f\right]\left(t,\omega\right)\, Z\left[\Im f\right]\left(t+\tfrac{1}{2},\omega\right) \tag{18}$$

for almost all $(t,\omega) \in \left[0,\frac{1}{2}\right) \times \left[0,1\right)$, where $\Re z$ and $\Im z$ denote, respectively, the real and imaginary part of $z \in \mathbb{C}$.

We can restate this result in the more geometric language of Theorem 25 from [17] saying that if $\mathcal{T}$ is a mapping from $L^2(\mathbb{R})$ into $L^2([0,\frac{1}{2})\times[0,1),\mathbb{C}^2)$ defined by

$$\mathcal{T}f(t,\omega) = \left(Z\,f\left(t,\omega\right), Z\,f\left(t+\tfrac{1}{2},\omega\right)\right),$$

then f is an atom of the Wilson orthonormal basis if and only if the points $(0,0)$, $\mathcal{T}[\Re f](t,\omega)$, $\mathcal{T}[\Im f](t,\omega)$ are *colinear* for almost all $(t,\omega) \in \left[0,\frac{1}{2}\right) \times \left[0,1\right)$. Note that, in the similar language, the atoms $f \in \mathcal{S}$ are characterized by the property that the points $\mathcal{T}f(t,\omega)$ lie on the unit sphere S^1 in $\mathbb{C}^2$ for almost all $(t,\omega) \in \left[0,\frac{1}{2}\right) \times \left[0,1\right)$, where

$$S^1 = \{(z,w) \in \mathbb{C}^2 : |z|^2 + |w|^2 = 1\}.$$

Proof of Proposition 4.3. Let $t \in \left[0,\frac{1}{2}\right]$, $\omega \in \left[0,1\right]$. Then the characterization condition (14) turns into

$$Z\,[MJf]\,(t,\omega)\ \overline{Zf(t,\omega)}\ +\ Z\,[MJf]\,(t+\tfrac{1}{2},\omega)\ \overline{Zf(t+\tfrac{1}{2},\omega)}\ =\ 0.$$

Since the image of f under the operator M is mapped by Zak transform to

$$Z[Mf]\big(t,\omega\big) = e^{2\pi i t} Zf\big(t,\omega\big), \tag{19}$$

reasoning as in Proposition 3.1, we obtain

$$Zf(t,\omega)\ \overline{Zf(t+\tfrac{1}{2},1-\omega)}\ =\ Zf(t+\tfrac{1}{2},\omega)\ \overline{Zf(t,1-\omega)}. \tag{20}$$

Using the identity $Zf(t,1-\omega)\ =\ \overline{Z\,\overline{f}\,(t,\omega)}$, we obtain

$$Zf(t,\omega)\ \ Z\,\overline{f}\,(t+\tfrac{1}{2},\omega)\ =\ Z\,\overline{f}\,(t,\omega)\ \ Zf(t+\tfrac{1}{2},\omega). \tag{21}$$

Now since Z is linear and for any complex numbers z,w it holds that

$$z\,\overline{w}\ =\ \overline{z}\,w\ \Leftrightarrow\ \Im z\,\Re w\ =\ \Re z\,\Im w,$$

one gets the equivalence of (21) to (18). $\square$

5 Other Modified Systems

There are two remaining modified Wilson systems, namely:

1. the system composed of $(M^m f)_{m\in\mathbb{Z}}$ and

$$\left[2^{-1/2}(M^m T^n f + (-1)^n M^m T^{-n} f)\right]_{n\geq 1,\, m\in\mathbb{Z}}, \tag{22}$$

2. the system composed of $(M^m f)_{m\in\mathbb{Z}}$ and

$$\left[2^{-1/2}(M^m T^n f + M^m T^{-n} f)\right]_{n\geq 1,\, m\in\mathbb{Z}}. \tag{23}$$

Replacing J with MTJ and TJ, we obtain the analogues of identity (7):

$$\begin{aligned}(MTJ)\,M^mT^n \;&=\; M^m(-T)^{-n}\,(MTJ),\\ (TJ)\,M^mT^n \;&=\; M^mT^{-n}\,(TJ),\end{aligned}$$

which by the reasoning similar to this in the previous sections give the characterizations in terms of inner products, of E_n functions, and of Zak transforms.

Corollary 5.1 *Let $f \in \mathcal{S}$. Then the Wilson system (22), respectively (23), is an orthonormal basis in $L^2(\mathbb{R})$ if and only if*

$$\forall_{m,n\in\mathbb{Z}} \quad \langle MTJf, M^{2m}T^{2n}f\rangle = 0, \tag{24}$$

respectively

$$\forall_{m,n\in\mathbb{Z}} \quad \langle TJf, M^{2m}T^{2n}f\rangle = 0, \tag{25}$$

holds.

Corollary 5.2 *The condition (24), respectively (25), is equivalent to*

$$E_n(x) = \sum_k (-1)^k\, f(x-k)\,\overline{f(x-2n+k)} \;=\; 0,$$

respectively,

$$E_n(x) = \sum_k \; f(x-k)\,\overline{f(x-2n+k)} \;=\; 0$$

for almost all $x \in [0,1]$ and for any $n \in \mathbb{Z}$.

Corollary 5.3 *The conditions (24), respectively (25), are equivalent to*

$$e^{-2\pi i\omega}\, Zf(t,1-\omega)\,\overline{Zf(t,\omega)} \;+\; Zf\big(t+\tfrac{1}{2},1-\omega\big)\,\overline{Zf(t+\tfrac{1}{2},\omega)} \;=\; 0$$

and

$$e^{-2\pi i\omega}\, Zf(t,1-\omega)\,\overline{Zf(t,\omega)} \;-\; Zf\big(t+\tfrac{1}{2},1-\omega\big)\,\overline{Zf(t+\tfrac{1}{2},\omega)} \;=\; 0$$

for almost all $(t,\omega) \in \big[0,\tfrac{1}{2}\big) \times \big[0,1\big)$, respectively.

Proof follows from the combination of the below properties of Zak transform and of operator J:

$$Z[TJf]\big(t,\omega\big) = e^{-2\pi i\omega}Zf(t,1-\omega), \quad Z[MTJf]\big(t,\omega\big) = e^{2\pi it}e^{-2\pi i\omega}Zf(t,1-\omega),$$

$$Z[TJf]\big(t+\tfrac{1}{2},\omega\big) = Zf\big(t+\tfrac{1}{2},1-\omega\big), \quad Z[MTJf]\big(t+\tfrac{1}{2},\omega\big) = -e^{2\pi it}Zf\big(t+\tfrac{1}{2},1-\omega\big).$$

□

6 Complementary Systems

The complementary Wilson system has the same properties as the original one. The results of the paper, in particular, Theorems 2.1, 4.1, and Corollaries 5.1, 5.2 remain true for the complementary Wilson systems such as (2) and the systems defined as:

1. the system composed of $(M^{2m+1}f)_{m\in\mathbb{Z}}$ and
$$\left[2^{-1/2}(M^mT^nf-(-1)^mM^mT^{-n}f)\right]_{n\geq 1,\, m\in\mathbb{Z}}\quad , \tag{26}$$
2. the system composed of $(M^mf)_{m\in\mathbb{Z}}$ and
$$\left[2^{-1/2}(M^mT^nf-(-1)^nM^mT^{-n}f)\right]_{n\geq 1,\, m\in\mathbb{Z}}\quad , \tag{27}$$
3. the system composed of
$$\left[2^{-1/2}(M^mT^nf-M^mT^{-n}f)\right]_{n\geq 1,\, m\in\mathbb{Z}}\quad . \tag{28}$$

For instance, in the case of the system (26) we obtain that

$$2W-S=-\sum\left\langle\,\cdot\,,\,(-M)^mT^{-n}f\right\rangle\; M^mT^nf,$$

which is null if and only if (6) holds by exactly the same argument as in the proof of Theorem 2.1.

Corollary 6.1 *Let $f\in\mathcal{S}$. Then the Wilson system (2), respectively (26), (27), (28), is an orthonormal basis in $L^2(\mathbb{R})$ if and only if (14), respectively (6), (24), (25), holds.*

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